Centralizer Coalgebras, FRT-Construction and Symplectic Monoids

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1 Introduction

L. Faddeev, N. Reshetikhin und L. Takhtadjian [RTF] introduced a construction to obtain quantum deformations of coordinate rings of classical groups. General considerations about this so called FRT-construction can be found for instance in [Ma],[Ta], [Ha], [Su] and also in many textbooks on quantumgroups. Our approach differs from former ones in the following three aspects:

First, we focus attention to the *graded matric bialgebra* which arises in the first step of the construction. This means that we rather look at quantizations of appropriate closed monoids instead of classical groups. Especially we look at the *homogenous summands* of these graded bialgebras. These are coalgebras which can be defined in a dual way to *centralizer algebras* of subsets in an endomorphism ring. We therefore call them *centralizer coalgebras* and investigate their relationship to the corresponding *centralizer algebras*.

Further, we work over arbitrary noetherian integral domains as base rings. This makes sense since all known examples are allready welldefined over rings of integral *Laurent* polynomials, i.e. $\mathbb{Z}[X, X^{-1}]$ (in the indeterminant X). We will see that there are tremendous differences to the theory over fields, especially concerning the comparison of centralizer algebras and coalgebras. For instance the centralizer coalgebra may have R-torsion. We present the following criterion for R-projectivity: This property holds if and only if the centralizer algebra is stable under base changes. Furthermore, we will see that the latter property is allways valid for centralizer coalgebras.

Finally, the FRT-construction in the ordinary form depends on exactly one endomorphism which usually is a *quantum Yang-Baxter operator* in the applications. Here we give another version of the FRT-construction which can be applied to sets of endomorphisms. This generalization is neccessary to describe the coordinate rings of classical symplectic and orthogonal monoids by use of an FRT-type construction. We demonstrate this in the symplectic case, giving some improvements of results by S. Doty [Dt]. Furthermore, applying our results on centralizer coalgebras we obtain an integral form for the *symplectic Schur algebra* defined by S. Donkin in [Do2] without any use of the hyperalgebra or Kostant $\mathbb{Z}$-form. As an additional incredience we need a symplectic version of the straightening formula for bideterminants the proof of which covers all of the last two sections.

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2 Centralizer Coalgebras

Let R be a noetherian integral domain, V a free R-module with a fixed basis $\{v_1, \ldots, v_n\}$. Let e_i^j denote the corresponding basis of matrix units for $\mathcal{E} := \mathrm{End}_R(V)$. The algebra structure on $\mathcal{E}$ induces a coalgebra structure on the dual R-module $\mathcal{E}^* := \mathrm{Hom}_R(\mathcal{E}, R)$. On the dual basis elements the comultiplication Δ and the counit ϵ are given by

$$\Delta(e_i^{*j}) = \sum_{k=1}^{n} e_i^{*k} \otimes e_k^{*j}, \quad \epsilon(e_i^{*j}) = \delta_{ij}.$$

We will be engaged with epimorphic coalgebra images of $\mathcal{E}^*$ since they correspond to subalgebras of $\mathcal{E}$. Consider the isomorphism $\vartheta_{tr} : \mathcal{E} \to \mathcal{E}^*$ of R-modules induced by the matrix trace map $tr : \mathcal{E} \to R$ or more precisely by the corresponding nondegenerated bilinear form. Let $A \subseteq \mathcal{E}$ be an arbitrary subset and $L(A)$ the R-linear span in $\mathcal{E}$ of all commutators $[\nu, \mu] = \nu\mu - \mu\nu$ where ν runs through A and μ runs through $\mathcal{E}$. We set $K(A) := \vartheta_{tr}(L(A)) \subseteq \mathcal{E}^*$. If A consists of just one element ν we use abbreviations $L(\nu)$ and $K(\nu)$ for $L(A)$ and $K(A)$. The proofs of the following lemmata are straightforward and can be found in section 1.3 of [Oe].

Lemma 2.1 *$K(A)$ is a coideal in $\mathcal{E}^*$ for each subset $A \subseteq \mathcal{E}$.*

If C is an arbitrary coalgebra such that V is a C-comodule with structure map $\tau_V : V \to V \otimes C$ we denote the set of C-comodule endomorphisms by

$$\mathrm{End}_C(V) := \{\mu \in \mathcal{E} | \ (\mu \otimes \mathrm{id}_C) \circ \tau_V = \tau_V \circ \mu\}.$$

Clearly this is a subalgebra of $\mathcal{E}$.

Lemma 2.2 *Let C be a coalgebra together with an epimorphism $\pi : \mathcal{E}^* \to C$. Then*

$$\mu \in \mathrm{End}_C(V) \Longleftrightarrow K(\mu) \subseteq \ker(\pi).$$

Corollary 2.3 *Let M be a coideal in $\mathcal{E}^*$ and $\mu, \nu \in \mathcal{E}$. Then $K(\mu) \subseteq M$ and $K(\nu) \subseteq M$ implies $K(\mu\nu) \subseteq M$ and $K(\nu\mu) \subseteq M$.*

We now define the *centralizer coalgebra* of the subset $A \subseteq \mathcal{E}$ as

$$M(A) := \mathcal{E}^*/K(A).$$

According to lemma 2.2 we have

$$A \subseteq \mathrm{End}_{M(A)}(V).$$

Furthermore $M(A)$ is the largest epimorphic image of $\mathcal{E}^*$ with this property. By the corollary $M(A)$ does not change if A is substituted by its algebraic span. Before going deeper into the analysis of relationships between $M(A)$ and the *centralizer algebra*

$$C(A) := \mathrm{End}_A(V) = \{\mu \in \mathcal{E} | \ [\mu, \nu] = 0, \quad \text{for all } \nu \in A\}$$

we give a presentation of $M(A)$ by generators and relations, which is convenient for practical use.

The residue classes of the basis elements e_i^{*j} with respect to any coideal in $\mathcal{E}^*$ will allways be denoted by x_{ij} where $i,j \in \underline{n} := \{1,\ldots,n\}$. If $\mu = \sum_{i,j=1}^n a_{ij}e_i^j \in \mathcal{E}$ is arbitrary we write

$$\mu x_{ij} := \sum_{k=1}^n a_{ik}x_{kj} \quad \text{und} \quad x_{ij}\mu := \sum_{k=1}^n x_{ik}a_{kj}. \tag{1}$$

Now, if N is a subset of $\mathcal{E}$ and A its algebraic span then $M(A)$ is defined by the relations

$$\mu x_{ij} = x_{ij}\mu \quad \text{for all } \mu \in N,\ i,j \in \underline{n}. \tag{2}$$

As consequences one has relations of the same form where μ runs through all of A.

3 Comparison Theorems

Remember the definition of the *complement*

$$U^\perp := \{f \in W^* = \mathrm{Hom}_R(W,R) |\ f(u) = 0\ \forall u \in U\}$$

of a submodule U in an R-module W and in the definition of the *evaluation map*

$$\mathrm{Ev}_W : W \to W^{**}, \quad \text{given by } \mathrm{Ev}_W(x)(y) := y(x)\ x \in W,\ y \in W^*$$

In W^{***} we have the following commutatvity rule

$$\mathrm{Ev}_{W^*}(U^\perp) = \mathrm{Ev}_W(U)^\perp. \tag{3}$$

Turning to our special situation we first note that $tr(b[x,a]) = tr(x[a,b])$ for all $x,a,b \in \mathcal{E}$. Since the bilinearform induced by tr is nondegenerated, it follows:

$$\vartheta_{tr}([x,a])(b) = tr(b[x,a]) = 0 \text{ for all } x \in \mathcal{E} \Longleftrightarrow [a,b] = 0. \tag{4}$$

The following fundamental lemma of this section is easy to prove now.

Lemma 3.1 *We have* $K(A)^\perp = \mathrm{Ev}_{\mathcal{E}}(C(A))$ *and* $K(A)^{\perp^\perp} = \mathrm{Ev}_{\mathcal{E}^*}(C(A)^\perp)$

Proof: According to the definition $\mathrm{Ev}_{\mathcal{E}}(b) \in K(A)^\perp$ if and only if $\mathrm{Ev}_{\mathcal{E}}(b)(\vartheta_{tr}([x,a])) = \vartheta_{tr}([x,a])(b) = 0$ for all $x \in \mathcal{E}$ and $a \in A$. Applying (4) this is the case if and only if $[a,b] = 0$ for all $a \in A$, thus if and only if $b \in C(A)$. The second equation follows from the first by use of equation (3). □

Remember that the dual module $C^* = \mathrm{Hom}_R(C,R)$ of a coalgebra C allways posseses the structure of an algebra by use of the *convolution product*

$$\mu\nu := (\mu \otimes \nu) \circ \Delta \quad \text{where } \mu,\nu \in C^*.$$

Here, we have identified $\mu\otimes\nu$ with its image under the natural homomorphism $C^*\otimes C^* \to (C \otimes C)^*$. Note that this construction is functorial. In the special case $C = \mathcal{E}^*$ we obtain an algebra structure on $\mathcal{E}^{**}$. Furthermore it is easy to show that the evaluation map $\mathrm{Ev}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{**}$ is an isomorphism of algebras.

Lemma 3.2 *Let C be a coalgebra together with an epimorphism $\pi : \mathcal{E}^* \to C$. Set $K := \ker(\pi)$. Then $\rho := \mathrm{Ev}_{\mathcal{E}}^{-1} \circ \pi^* : C^* \to \mathcal{E}$ is a monomorphism of R-algebras and* $\mathrm{im}(\rho) = \mathrm{Ev}_{\mathcal{E}}^{-1}(K^{\perp})$.

PROOF: By functoriality the dual map $\pi^* : C^* \to \mathcal{E}^{**}$ is an algebra homomorphism. Since $\mathrm{Hom}_R(-, R)$ is exact on the right, π^* is injective. One easily shows $\mathrm{im}(\pi^*) = K^{\perp}$. This completes the proof, since $\mathrm{Ev}_{\mathcal{E}}$ is an algebra isomorphism as mentioned above. □

Theorem 3.3 (1. Comparison Theorem) *The centralizer algebras $C(A)$ and the dual of the centralizer coalgebra $M(A)^*$ are isomorphic to each other. An isomorphism is given by ρ.*

PROOF: This follows immediately from lemma 3.1 and 3.2. □

Now, let us compare the dual of $C(A)$ with $M(A)$. To this aim we consider the dual map $J^* : \mathcal{E}^* \to C(A)^*$ of the inclusion $J : C(A) \hookrightarrow \mathcal{E}$. Because of $\mathrm{Ev}_{\mathcal{E}^*}(U) \subseteq U^{\perp\perp}$ for arbitrary submodules U and according to Lemma 3.1 we have

$$K(A) \subseteq \mathrm{Ev}_{\mathcal{E}^*}^{-1}(K(A)^{\perp\perp}) = \mathrm{Ev}_{\mathcal{E}^*}^{-1}(\mathrm{Ev}_{\mathcal{E}^*}(C(A)^{\perp})) = C(A)^{\perp} = \ker(J^*).$$

Therefore J^* factors to an R-modul homomorphism

$$\theta : M(A) \to C(A)^*.$$

Lemma 3.4 *The kernel of θ is precisely the torsion submodule of $M(A)$*

PROOF: From general results of commutative algebra (cf. [Oe] Anhang A 1.2) it follows that the torsion submodule of $M(A)$ coincides with $\mathrm{Ev}_{\mathcal{E}^*}^{-1}(K(A)^{\perp\perp})/K(A)$. By the above calculations this is just the kernel $C(A)^{\perp}/K(A)$ of θ. □

This immediately implies

Corollary 3.5 (Criterion of Torsionfreeness) *The following statements are equivalent:*

(a) $M(A)$ *is torsionfree*

(b) $K(A) = C(A)^{\perp}$

(c) θ *is injective.*

Remark 3.6 *The map θ is surjective if and only if the extensiongroup $\mathrm{Ext}_R^1(\mathcal{E}/C(A), R)$ is trivial, in particular if $C(A)$ is a direct summand in $\mathcal{E}$.*

At the beginning of this section we have constructed an algebra structure on the dual module of a coalgebra. Conversely we should obtain a coalgebra structure on the dual of an algebra A. Whereas this is not possible in general, it can be done under certain restrictions to the R-module structure of the algebra A. To be more precise, the natural R-homomorphism from $A^* \otimes A^*$ into $(A \otimes A)^*$ must be an isomorphism. This is the case if A is projective and finetely generated. Therefore, under this circumstances the construction is allways possible and functorial, that is: duals of algebra maps become coalgebra maps. Thus we obtain

Theorem 3.7 (2. Comparison Theorem) *Suppose the centralizer algebra $C(A)$ of A is a direct summand in $\mathcal{E}$ as an R-module. Then $C(A)^*$ can be turned into a coalgebra. The map $\theta : M(A) \to C(A)^*$ is an epimorphism of coalgebras, whose kernel is just the torsion submodule of $M(A)$.*

4 Change of Base Rings

Let S be another noetherian integral domain together with a homomorphism $R \to S$ of rings. We will consider S as an R-algebra via this homomorphism. In this situation there is a functor from the category of R-modules into the category of S-modules given by

$$W^S := S \otimes_R W \quad \text{und} \quad \alpha^S := \mathrm{id}_S \otimes \alpha : U^S \to W^S \tag{5}$$

to each pair of R-moduls W and U and an R-homomorphism $\alpha : U \to W$. We will study the behaviour of the construction of centralizer coalgebras under these functors. It will turn out that the centralizer coalgebra behaves better than the centralizer algebra. To start with let

$$\mathcal{E}_S := \mathrm{End}_S(V^S), \;\; \zeta_S : \mathcal{E}^S \to \mathcal{E}_S$$

where ζ_S is given by $\zeta_S(s \otimes e)(t \otimes v) := st \otimes e(v)$ for all $s, t \in S, e \in \mathrm{End}_R(V), v \in V$. Let $J_A : A \hookrightarrow \mathcal{E}$ be the natural embedding of the subalgebra $A \subseteq \mathcal{E}$ and let

$$A_S := \mathrm{im}(\zeta_S \circ {J_A}^S)$$

denote the image of A^S in $\mathcal{E}_S$. Further there are natural homomorphisms

$$\eta_S : C(A)^S \to C(A_S) = \mathrm{End}_{A_S}(V^S) \subseteq \mathcal{E}_S$$

on generators given in a similar way to ζ_S. Note, that both, ζ_S and η_S are homomorphisms of algebras connected by the equation

$$J_{C(A_S)} \circ \eta_S = \zeta_S \circ {J_{C(A)}}^S.$$

Thus η_S is injective if and only if ${J_{C(A)}}^S$ is injective (J stands for the corresponding embeddings). This may fail if $C(A)$ is not a pure R-submodule in $\mathcal{E}$. Further η_S may fail to be surjective (see the example following theorem 4.3). $C(A)$ is called *stable under base change* if η_S is an isomorphism of for all choices for S. Now, in analogy to η_S we are going to consider natural homomorphisms

$$\mu_S : M(A)^S \to M(A_S) = (\mathcal{E}_S)^*/K(A_S)$$

We will show that they are isomorphisms independent on the choices for A, R and S. To this claim we consider

$$\chi_S := {\zeta_S}^{*-1} \circ \psi_{\mathcal{E}} : \mathcal{E}^{*S} \to (\mathcal{E}_S)^*$$

where $\psi_{\mathcal{E}} : \mathcal{E}^{*S} \to \mathcal{E}^{S*}$ is the natural homomorphism given by $\psi_{\mathcal{E}}(s \otimes f)(t \otimes e) := stf(e)$ on genorators with $s, t \in S,\ f \in \mathcal{E}^*\ e \in W$. It is easy to check that χ_S is a homomorphism of coalgebras if the coalgebra structure on $\mathcal{E}^{*S}$ is defined in a canonical way (for details see [Oe] section 1.5). Further, the verification of the commutativity rule

$$\chi_S \circ {\vartheta_{tr}}^S = \vartheta_{tr_S} \circ \zeta_S \tag{6}$$

is straightforward, as well. Here $\vartheta_{tr_S} : \mathcal{E}_S \to (\mathcal{E}_S)^*$ is the isomorphism induced by the matrix trace map $tr_S : \mathcal{E}_S \to S$. Setting

$$L(A_S) :=< [\nu, \mu] | \nu \in A_S, \mu \in \mathcal{E}_S >_{S-\mathrm{mod}},$$

we obtain

$$\mathrm{im}(\zeta_S \circ {J_{L(A)}}^S) = L(A_S),$$

since ζ_S is an isomorphism of S-algebras. By definition we have $K(A)_S = \vartheta_{tr_S}(L(A)_S)$ and $\mathrm{im}({J_{K(A)}}^S) = \mathrm{im}({\vartheta_{tr}}^S \circ {J_{L(A)}}^S)$. Using (6) this yields

$$\mathrm{im}(\chi_S \circ {J_{K(A)}}^S) = K(A_S). \tag{7}$$

Here again, we have used the symbol J to indicate embeddings of R-submodules. Note that in particular $K := \mathrm{im}({J_{K(A)}}^S)$ is a coideal in $\mathcal{E}^{*S}$ since χ_S is an isomorphism of coalgebras and therefore $M(A)^S \cong \mathcal{E}^{*S}/K$ is a coalgebra. Finally, we are able to define the natural homomorphism μ_S as the factorization of χ_S which exists by (7). We immediately obtain

Theorem 4.1 (Change of Base Rings) *For any noetherian integral domain S which is an R-algebra and any R-subalgebra A of $\mathcal{E}$ there is a natural homomorphism*

$$\mu_S : S \otimes_R M(A) \to M(S \otimes_R A)$$

which is an isomorphism of S-coalgebras.

This means that $M(A)$ is stable under base changes for all choices of A and R. If the R-algebra S is a field, it follows from theorem 3.3 and 4.1 that

$$\dim_S(C(A_S)) = \dim_S(M(A_S)^*) = \dim_S((M(A)^S)^*) = \dim_S(M(A)^S) \tag{8}$$

Now, for a noetherian integral domain R it is known from commutative algebra that an R-module W is projective if and only if the dimension of W^S is independent on the field S. Thus we obtain

Corollary 4.2 *Let F be the field of fractions of R. Then $M(A)$ is projective if and only if $\dim_S(C(A_S)) = \dim_F(C(A_F))$ holds for each field S.*

Theorem 4.3 (Criterion of Projectivity) *The following statements are equivalent:*

(a) The centralizer coalgebra $M(A)$ ist projektiv.

(b) The centralizer algebra $C(A) = \mathrm{End}_A(V)$ is stable under base changes.

(c) $M(A)$ is torsionfree and $C(A)$ a direct summand in $\mathcal{E}$.

PROOF: First assume (a). Then the sequence

$$0 \to K(A) \to \mathcal{E}^* \to M(A) \to 0$$

is split and consequentely the same is true for

$$0 \to M(A)^* \to \mathcal{E}^{**} \to \mathcal{E}^{**}/K(A)^\perp \to 0.$$

Since $\mathrm{Ev}_\mathcal{E}$ induces an isomorphism between $\mathcal{E}^{**}/K(A)^\perp$ and $\mathcal{E}/C(A)$ according to lemma 3.1 it follows that $\mathcal{E}/C(A)$ is projective, as well. Thus $C(A)$ is a direct summand in $\mathcal{E}$ proving (c).

Part (a) follows from (c) by theorem 3.7, since the dual of a projective module is projective again. To verify (b) we therefore may assume both (a) and (c). Since $C(A)$ is a direct summand ${J_{C(A)}}^S$ is injective for all R-algebras S. Consequentely all η_S are injective (see above). To show surjectivity note that the image $\mathrm{im}(\zeta_S \circ {J_{C(A)}}^S) = \mathrm{im}(J_{C(A_S)} \circ \eta_S)$ of ${C(A)}^S$ in $\mathcal{E}_S$ must be a direct summand therein, since ζ_S is an isomorphism and $\mathrm{im}({J_{C(A)}}^S)$ a direkt summand in $\mathcal{E}^S$. Therefore, to show that this submodule of $\mathcal{E}_S$ coincides with $C(A_S)$, it is enought to verify that both have the same rank (the dimension of the G-tensored module over the field G of fractions on S). But these ranks must indeed be the same as can be seen from the following calculations

$$\dim_G({C(A)}^G) = \dim_F({C(A)}^F) = \dim_F(C(A_F)) = \dim_G(C(A_G)).$$

where the lefthandside equation holds by projectivity of $C(A)$, the righthandside one by corollary 4.2 and the one in the middle since η_F is an isomorphism by flatness of the field F of fractions on R. This establishes (b).

Now assume (b). This implies that the map ${J_{C(A)}}^S$ induced by the embedding $J_{C(A)}$ is injective for all S. By commutative algebra arguments one concludes that $C(A)$ is a direct summand in the R-free module $\mathcal{E}$, in particular it is projective. Now, let S be a field. Since η_S is an isomorphism we have

$$\dim_S({C(A)}^S) = \dim_S(C(A_S)).$$

The lefthandside is independent of S by projectivity of $C(A)$. Thus by corollary 4.2 $M(A)$ is projective yielding (a). $\square$

Example: Let $R = \mathbb{Z}$ and $V = \mathbb{Z}^4$. Further let

$$a := \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \in \mathcal{E} = \mathrm{End}_\mathbb{Z}(\mathbb{Z}^4)$$

and $A :=< a >$ be the subalgebra in $\mathcal{E}$ generated by a. Each field can be considered as a $\mathbb{Z}$-algebra. For a field of characteristic different from 2 the minimum polynomial of a^S is $t^2 - 2t$, but in characteristic 2 it is t^2. This means that the algebra A_S is two dimensional for each field S. It follows that A is a direct summand in $\mathcal{E}$. But nethertheless $M(A)$ is not projective (free), because in the case of a field of characteristic different from 2 a^S is diagonalisable and one calculates $\dim_S(C(A_S)) = 2^2 + 2^2 = 8$, while in the case of characteristic 2 we have $\dim_S(C(A_S)) = 9$ gilt. Therefore $M(A)$ can't be projective in view of corollary 4.2.

5 FRT-Construction

The r fold tensor product of V is denoted by $V^{\otimes r}$. Since it is a free R-module, as well, we may apply all results of former sections to a situation where V is substituted by $V^{\otimes r}$, $\mathcal{E}$ by $\mathcal{E}_r := \mathrm{End}_R(V^{\otimes r})$ and $\mathcal{E}^*$ by ${\mathcal{E}_r}^* := \mathrm{Hom}_R(\mathcal{E}_r, R)$. There are natural isomorphisms between $\mathcal{E}_r$ and $\mathcal{E}^{\otimes r}$ and between ${\mathcal{E}_r}^*$ and $\mathcal{E}^{*\otimes r}$. We will omit them in our notation and consider them as identity maps. Set $V^{\otimes 0} := \mathcal{E}_0 := R$.

Suppose we are given a family $\mathcal{A} = (\mathcal{A}_r)_{r\in\mathbb{N}_0}$ of R-algebras such that $\mathcal{A}_r \subseteq \mathcal{E}_r$. According to lemma 2.1 there is an associated family of coideals K_r in ${\mathcal{E}_r}^*$.

$$K_r := K(\mathcal{A}_r). \tag{9}$$

By the above identification we consider $\mathcal{T}(\mathcal{E}^*) := \bigoplus_{r\in\mathbb{N}_0} {\mathcal{E}_r}^*$ as the tensor algebra on the R-module $\mathcal{E}^*$. There is a unic coalgebra structure on this tensor algebra, extending the one from $\mathcal{E}^*$ and turning $\mathcal{T}(\mathcal{E}^*)$ into a bialgebra. Furthermore the ${\mathcal{E}_r}^*$ are subcoalgebras dual to the algebras $\mathcal{E}_r$, i.e. the coalgebra structure on ${\mathcal{E}_r}^*$ is the one considered above. Epimorphic images of this bialgebra are called *matric bialgebras* (cf. [Ta]). Let us investigate under what circumstances the coideal

$$I := \oplus_{r\in\mathbb{N}_0} K_r \tag{10}$$

becomes an ideal in $\mathcal{T}(\mathcal{E}^*)$ and thus consequentely is a biideal. To this claim we consider inclusion maps $s_r, t_r : \mathcal{E}_r \to \mathcal{E}_{r+1}$ given by

$$s_r(\mu) = \mu \otimes \mathrm{id}_V,\ t_r(\mu) = \mathrm{id}_V \otimes \mu,\ \mu \in \mathcal{E}_r$$

and focus attention to a special situation, which is general enought for our applications. We start with an arbitrary subset $N \subseteq \mathcal{E}_2$ and define a family $\mathcal{A}$ inductively beginning with $\mathcal{A}_0 := R,\ \mathcal{A}_1 := R \cdot \mathrm{id}_V$ and $\mathcal{A}_2$ as the algebraic span of N in $\mathcal{E}_2$, than continuing by the formula

$$\mathcal{A}_r := \langle s_{r-1}(\mathcal{A}_{r-1}) + t_{r-1}(\mathcal{A}_{r-1})\rangle_{\mathrm{Alg}} \quad \text{for} \quad r > 2.$$

We call this a *by N induced family* of subalgebras of $\mathcal{E}_r$.

Proposition 5.1 *Let $\mathcal{A}$ be the family induced by a subset $N \subseteq \mathcal{E}_2$. Then the coideal I defined in (10) is a homogenous biideal in $\mathcal{T}(\mathcal{E}^*)$ generated by the coideal K_2.*

Proof: First we show that I is a homogenous ideal. To this claim take $a \in K_r$ and $b \in {\mathcal{E}_u}^*$. We have to show $a \otimes b \in K_{r+u}$ and $b \otimes a \in K_{r+u}$. The choice of the element a can be reduced to a generator $a = \vartheta_{tr}([\mu,\nu])$ of the R-module K_r with $\mu \in \mathcal{E}_r$ and $\nu \in \mathcal{A}_r$. Letting $\hat{\nu} := s_{r+u-1} \circ s_{r+u-2} \circ \ldots \circ s_r(\nu) = \nu \otimes \mathrm{id}_{V^{\otimes u}} \in \mathcal{A}_{r+u}$ and $\hat{\mu} := \mu \otimes \bar{b}$, where $\bar{b} \in \mathcal{E}_u$ is the preimage of b under ϑ_{tr} one obtains

$$a \otimes b = \vartheta_{tr}([\mu,\nu]) \otimes b = \vartheta_{tr}([\hat{\mu},\hat{\nu}]) \in K_{r+u}.$$

Similary $b \otimes a \in K_{r+u}$ and the first statement is established.

For the second part denote by J the homogenous ideal in $\mathcal{T}(\mathcal{E}^*)$ generated by K_2. Since I is a homogenous ideal containing K_2 we get $J \subseteq I$. For the reverse inclusion we show $K_r \subseteq J$ by induction on r. Clearly $K_0 = K_1 = (0)$ and K_2 are contained in J. Now suppose $r > 2$. Letting $J_r := J \cap \mathcal{E}_r{}^*$ one obtains $J_r = J_{r-1} \otimes \mathcal{E}^* + \mathcal{E}^* \otimes J_{r-1}$. By induction hypothesis we see $K_{r-1} = J_{r-1}$. Therefore

$$J_r = K_{r-1} \otimes \mathcal{E}^* + \mathcal{E}^* \otimes K_{r-1} = K(M)$$

where $M := s_{r-1}(\mathcal{A}_{r-1}) + t_{r-1}(\mathcal{A}_{r-1}) \subseteq \mathcal{E}_r$. But since $\mathcal{A}_r$ is generated by M as an algebra we finally see from corollary 2.3 $K_r = K(\mathcal{A}_r) = K(M) = J_r$. □

According to the proposition we may assign a *graded matric bialgebra* to each subset $N \subseteq \mathcal{E}_2$ by

$$\mathcal{M}(N) := \mathcal{T}(\mathcal{E}^*)/I \tag{11}$$

whose homogenous summand are the centralizer coalgebras $M(\mathcal{A}_r)$. This bialgebra will be called the *FRT-construction* corresponding to the subset N. The reader familiar with the ordinary FRT-construction will recognize the latter one as the special case where N consists of just one element β (use the description (12) below). In this case we write $\mathcal{M}(\beta) := \mathcal{M}(N)$. Usually in the applications this β is a *quantum Yang-Baxter operator* leading to a representation of the *Artin braid groups* on the modules $V^{\otimes r}$. In this situaton the algebras $\mathcal{A}_r$ are just the images of the corresponding group algebras (over R) under this representations.

An application where N must consist of two elements will be given in the next section. Here the $\mathcal{A}_r$ are images of the *Brauer centralizer algebras* under Brauers representations corresponding to the symplectic groups. The bialgebra $\mathcal{M}(N)$ will turn out to be the coordinate ring of a certain *symplectic monoid*. It is a remarkable fact, that the second operator is only needed in the classical situation, whereas in the quantum case it lies in the algebraic span of the quantum Yang-Baxter operator (see [Oe], Bemerkung 2.5.1, 2.5.4). Thus, the ordinary FRT-construction behaves singular (in a certain sense) when specializing the deformation parameter to 1, i.e. in the classical limit.

We close this section giving are more convenient describtion of $\mathcal{M}(N)$. We denote by $I(n,r)$ the set of maps from von $\underline{r} := \{1,\ldots,r\}$ to $\underline{n} := \{1,\ldots,n\}$ and call the elements $\mathbf{i} \in I(n,r)$ multi-indices writing $\mathbf{i} = (i_1,\ldots,i_r)$, where $i_j \in \underline{n}$ for $j \in \underline{r}$. The residue classes of the multiplicative generators $e_i^{*j} + I$ of $\mathcal{M}(N)$ where $i,j \in \underline{n}$ will be denoted by x_{ij}. For pairs of multi-indices $\mathbf{i},\mathbf{j} \in I(n,r)$ we introduce the abbreviation

$$x_{\mathbf{ij}} := x_{i_1j_1}x_{i_2j_2}\ldots x_{i_rj_r}.$$

Using the notation introduced in (1) we obtain a presentation of $\mathcal{M}(N)$ by generators and relations given as follows

$$\mathcal{M}(N) = \langle x_{ij},\ i,j \in \underline{n} |\ \ \mu x_{\mathbf{ij}} = x_{\mathbf{ij}}\mu,\ \mathbf{i},\mathbf{j} \in I(n,2),\ \mu \in N \rangle\,. \tag{12}$$

The verification of this formula follows from the second statement of proposition 5.1 together with (2).

6 Example: Symplectic Monoids

Let $n = 2m$ be even. We will aply the FRT-construction to two endomorphisms $\beta,\gamma \in \mathcal{E}_2 = \mathcal{E} \otimes \mathcal{E}$. In order to define them we have to introduce some notation. First consider the involution $i' := n - i + 1$ on $\underline{n}$, that is

$$(1', 2', \ldots, n') = (n, n-1, \ldots, 1).$$

Further set $\epsilon_i := 1$ if $i \leq m$ and $\epsilon_i := -1$ if $i > m$ and define

$$\beta := \sum_{i,j \in \underline{n}} e_i^j \otimes e_j^i$$

$$\gamma := \sum_{i,j \in \underline{n}} \epsilon_i \epsilon_j e_i^{j'} \otimes e_{i'}^j.$$

The first endomorphism is just the flip operator on $V \otimes V$, whereas the second is an integer multiple of a projection map whose kernel is just the kernel of the linear form on $V \otimes V$ corresponding to the canonical skew bilinearform on V (see below) and whose image is the one dimensional span of a skew bivector. Our object of interest will be the FRT-construction

$$A_R^{\mathrm{s}}(n) := \mathcal{M}(\{\beta, \gamma\}).$$

According to the preceeding section it is a graded matric bialgebra whose homogenous summands

$$A_R^{\mathrm{s}}(n, r) := M(\mathcal{A}_r)$$

are the centralizer coalgebras of algebras $\mathcal{A}_r$ which are generated by endomorphisms

$$\beta_i := \mathrm{id}_{V^{\otimes i-1}} \otimes \beta \otimes \mathrm{id}_{V^{\otimes r-i-1}}, \in \mathcal{E}_r$$

$$\gamma_i := \mathrm{id}_{V^{\otimes i-1}} \otimes \gamma \otimes \mathrm{id}_{V^{\otimes r-i-1}} \in \mathcal{E}_r$$

for $i = 1, \ldots, r-1$. Using the notation of [We] the *Brauer centralizer algebra* $D_r(x)$ (x an element in R) is generated by symbols g_i and e_i for $i = 1, \ldots, r-1$ and the assignment $g_i \mapsto \beta_i$ and $e_i \mapsto \gamma_i$ defines a representation of $D_r(-n)$ on $V^{\otimes r}$. Thus $\mathcal{A}_r$ is just the image of $D_r(-n)$ under this representation, in H. Wenzl's notation from [We]: $\mathcal{A}_r = B_r(\mathrm{Sp}(n))$. R. Brauer showed in [Br] that this is just the centralizer algebra of the symplectic group $\mathrm{Sp}_{\mathbb{C}}(n)$ acting on $V^{\otimes r}$ if $R = \mathbb{C}$. One of our aims is to generalize this to the case of an arbitrary algebraically closed field K instead of $\mathbb{C}$.

Another problem, connected with the former, is to show that the centralizer algebra $C(\mathcal{A}_r)$ of $\mathcal{A}_r$ which will turn out to be the *symplectic Schur algebra* $S_0(n, r)$ defined by S. Donkin in [Do2] is stabel under base changes. In view of theorem 4.3 this is equivalent to the projectivity of the coalgebra $A_R^{\mathrm{s}}(n, r)$ as an R-module. For this purpose we are going to construct a basis for the latter one. The procedure follows [Oe] where the more general quantum case is treated. But it will become more transparent in the much simpler classical case. First note that there is an epimorphism of graded bialgebras from

$$A_R(n) := R[x_{11}, x_{12}, \ldots, x_{nn}]$$

to $A_R^{\mathrm{s}}(n)$ leaving the symbols x_{ij} fixed (we use x_{ij} as symbols for residue classes of e_i^{*j} in all cases of matric bialgebras). For $A_R(n)$ is just the FRT-construction $\mathcal{M}(\beta)$ where the relations coming from the endomorphism γ are omitted. This is because $x_{i_2 j_1} x_{i_1 j_2} =$

$\beta x_{\mathbf{ij}} = x_{\mathbf{ij}}\beta = x_{i_1 j_2} x_{i_2 j_1}$ just give the ordinary commutativity relations. The kernel of this bialgebra epimorphism is the ideal in $A_R(n)$ which is generated by the polynomials $\gamma x_{\mathbf{ij}} = x_{\mathbf{ij}}\gamma$ where $\mathbf{i}, \mathbf{j} \in I(n, 2)$. To write down these polynomials explicitely let us fix some notation:

$$f_{ij} := \sum_{k=1}^{n} \epsilon_k x_{ik} x_{jk'}, \quad \bar{f}_{ij} := \sum_{k=1}^{n} \epsilon_k x_{ki} x_{k'j} \in A_R(n, 2).$$

Setting $\mathbf{i} = (i, j)$ und $\mathbf{j} = (k, l)$ we obtain

$$\gamma x_{\mathbf{ij}} = \begin{cases} \epsilon_j \bar{f}_{kl} & i = j' \\ 0 & i \neq j' \end{cases} \qquad \text{und} \qquad x_{\mathbf{ij}}\gamma = \begin{cases} \epsilon_l f_{ij} & k = l' \\ 0 & k \neq l' \end{cases}$$

Therefore we have

$$A_R^{\mathrm{s}}(n) = A_R(n)/\mathcal{F} \tag{13}$$

where $\mathcal{F}$ is the ideal in $A_R(n)$ generated by the set

$$F := \{f_{ij}, \bar{f}_{ij}, f_{ll'} - \bar{f}_{kk'} |\ 1 \leq i < j \leq n, i \neq j',\ 1 \leq l \leq k \leq m\}. \tag{14}$$

If $R = K$ is an algebraically closed field we can interpret this in terms of algebraic geometry, i.e. we can look at the vanishing set of $\mathcal{F}$ in the monoid $\mathrm{M}_K(n)$ of $n \times n$-matrices. It is easy to see that this is just the closed submonoid

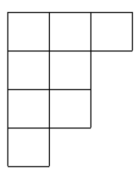
$$\mathrm{SpM}_K(n) := \{A \in \mathrm{M}_K(n) |\ \exists\ \ d(A) \in K,\ A^t J A = A J A^t = d(A) J\}$$

in $\mathrm{M}_K(n)$ called the *symplectic monoid* by S. Doty [Dt] and which has been considered by D.J. Grigor'ev [Gg] first. Here J is the Gram-matrix of the canonical skew bilinear form, that is $J = (J_{ij})_{i,j \in \underline{n}}$ where $J_{ij} := \epsilon_i \delta_{ij'}$. The function $d : \mathrm{SpM}_K(n) \to K$ is called the *coefficient of dilation*. It is neccesarily a regular function on $\mathrm{SpM}_K(n)$ and allready well defined in $A_K^{\mathrm{s}}(n)$, explicitely:

$$d = \epsilon_k f_{kk'} = \epsilon_k \bar{f}_{kk'} \in A_R^{\mathrm{s}}(n, 2). \tag{15}$$

Note that this is independet on $k \in \underline{n}$ by the relations in $A_K^{\mathrm{s}}(n)$. Furthermore d is a grouplike element of this bialgebra (cf. [Oe] 2.1.1). The set $\mathrm{GL}_K(n) \cap \mathrm{SpM}_K(n)$ of invertible elements in $\mathrm{SpM}_K(n)$ is precisely the group $\mathrm{GSp}_K(n)$ of *symplectic similitudes*. S. Doty showed in [Dt] that $\mathrm{SpM}_K(n)$ infact coincides with the Zariski-closure of $\mathrm{GSp}_K(n)$ in $\mathrm{M}_K(n)$. We will obtain this as an easy consequence of the results presented below.

To write down a basis for $A_R^{\mathrm{s}}(n, r)$ we need some combinatorics. The set of partitions of r is denoted by $\Lambda^+(r)$. It contains subsets $\Lambda^+(l, r)$ which consist of partitions having not more than l parts. We write partitions as l-tuples $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of nonnegative integers λ_i in descending order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 0$ such that $\lambda_1 + \ldots + \lambda_l = r$. To each partition one associates a *Young*-diagram reading row lenghts out of the components λ_i. For example



is associated to $\lambda = (3,2,2,1) \in \Lambda^+(4,8)$. The column lenghts of the diagram lead to another partition $\lambda' \in \Lambda^+(\lambda_1, r)$ called the dual of the partition λ, i.e. $\lambda'_i := |\{j|\ \lambda_j \geq i\}|$. Let $\mathcal{S}_r$ denote the symmetric group on r symbols and $\mathcal{S}_\lambda$ the *standard Young subgroup* of $\mathcal{S}_r$ corresponding to the partition λ. This is the subgroup fixing the subsets $\{1,\ldots,\lambda_1\},\ \{\lambda_1+1,\ldots,\lambda_1+\lambda_2\},\ldots$ of $\underline{r}$. In the above example $\lambda = (3,2,2,1)$ the standard Young subgroup of $\mathcal{S}_8$ corresponding to the dual partition λ' fixes $\{1,2,3,4\},\ \{5,6,7\}$ und $\{8\}$.

To each partition $\lambda \in \Lambda^+(r)$ and a pair of multi-indices $\mathbf{i},\mathbf{j} \in I(n,r)$ one defines a *bideterminant* $T^\lambda(\mathbf{i}:\mathbf{j}) \in A_R(n,r)$ by

$$T^\lambda(\mathbf{i}:\mathbf{j}) := \sum_{w\in\mathcal{S}_{\lambda'}} \mathrm{sign}(w) x_{\mathbf{i}(\mathbf{j}w)} = \sum_{w\in\mathcal{S}_{\lambda'}} \mathrm{sign}(w) x_{(\mathbf{i}w)\mathbf{j}}.$$

where $\mathbf{i}w := (i_{w(1)}, i_{w(2)}, \ldots, i_{w(r)})$. These are products of minor determinants, one factor for each column, the size of which correspond to the lenght of the column. By (13) they can be interpreted as elements of $A^{\mathrm{s}}_R(n,r)$, as well. We whish to write down a basis of the latter R-module consisting of such bideterminant. Since they are to large in number one needs a criterion to single out the right ones. This can be done using λ-tableaux. These are constructed from the diagram of λ by inserting the components of a multi-index column by column into the boxes. In the above example:

$$T^\lambda_{\mathbf{i}} := \begin{array}{|c|c|c|} \hline i_1 & i_5 & i_8 \\ \hline i_2 & i_6 & \\ \cline{1-2} i_3 & i_7 & \\ \cline{1-2} i_4 & & \\ \cline{1-1} \end{array} \quad .$$

We put a new order $\ll$ on the set $\underline{n}$, namely $1' \ll 1 \ll 2' \ll 2 \ll \ldots \ll m' \ll m$. A multi-index $\mathbf{i}$ is called λ-*column standard* if the entries in $T^\lambda_{\mathbf{i}}$ are strictly increasing down columns according to this order. It is called λ-*row standard* if the entries in $T^\lambda_{\mathbf{i}}$ are weakly increasing along rows and λ-*standard* if it is both at the same time. We write I_λ to denote the subset of $I(n,r)$ consisting of all multi-indices being λ-standard. Such a multi-index $\mathbf{i} \in I_\lambda$ is called λ-*symplectic standard* if for each index $i \in \underline{m}$ the occurences of i as well as i' in $T^\lambda_{\mathbf{i}}$ is limitted to the first i rows. The corresponding subset of I_λ will be denoted by I^{sym}_λ.

The notion of *symplectic standard tableaux* traces back to R.C. King [Ki] and it has appeard in a lot of work concerning symplectic groups and their representation theory (for details see [Do3]).

It is well known from invariant theory (cf. [Mr], section 2.5) that the collection of all bideterminants $T^\lambda(\mathbf{i}:\mathbf{j})$ where λ runs through $\Lambda^+(n,r)$ and $\mathbf{i},\mathbf{j}$ run through I_λ form a basis of $A_R(n,r)$. Similarily we will prove in the next section

Theorem 6.1 *The R-module $A^{\mathrm{s}}_R(n,r)$ has a basis given by*

$$\mathbf{B}_r := \{d^l T^\lambda(\mathbf{i}:\mathbf{j})|\ 0 \leq l \leq \frac{r}{2},\ \lambda \in \Lambda^+(m, r-2l),\ \mathbf{i},\mathbf{j} \in I^{\mathrm{sym}}_\lambda\}.$$

Before proving this let us have a look at some consequences. The first one generalizes theorem 9.5 (a) of [Dt] avoiding the restriction to characteristic zero. Furthermore, it contains corollary 5.5 (f) of that paper for allgebraically closed fields.

Corollary 6.2 *Let K be an algebraically close field. Then $A^{\mathrm{s}}_K(n)$ coincides with the coordinatering of the Zariski-closure $\overline{\mathrm{GSp}_K(n)}$ of $\mathrm{GSp}_K(n)$ in $\mathrm{M}_K(n)$. In particular $\mathrm{SpM}_K(n)$ is identical to $\overline{\mathrm{GSp}_K(n)}$. Therefore, a complete set of generators of the vanishing ideal of $\overline{\mathrm{GSp}_K(n)}$ in $A_K(n)$ is given by the set F (defined in equation 14).*

PROOF: Let $A_0(n)$ be the coordinate ring of $\overline{\mathrm{GSp}_K(n)}$ and $A_0(n,r)$ its r-th homogenous summand. In [Do2] the symplectic Schur algebra $S_0(n,r)$ is defined as the dual algebra to the coalgebra $A_0(n,r)$. The dimension of the latter one is given by Weyl's character formula and therefore independent on the field K (cf. [Do2] p. 77). On the other hand there is an epimorphism of graded bialgebras from $A^{\mathrm{s}}_K(n)$ to $A_0(n)$ since $\overline{\mathrm{GSp}_K(n)}$ is closed in $\mathrm{SpM}_K(n)$ and the latter one has been defined as the vanishing set of the ideal $\mathcal{F}$ by which $A^{\mathrm{s}}_K(n)$ is defined. But by our basis theorem 6.1 the dimension of $A^{\mathrm{s}}_K(n,r)$ is independent on the field K, as well. Thus, the proof can be finished looking at the case $K = \mathbb{C}$ and using Doty's theorem 9.5 (a) or alternately by a direct calculation of $|\mathbf{B}_r| = \dim_{\mathbb{C}}((A_0(n,r)))$ (see proposition 7.1 below). □

By theorem 4.1 we have isomorphisms

$$K \otimes_{\mathbb{Z}} A^{\mathrm{s}}_{\mathbb{Z}}(n,r) \cong A^{\mathrm{s}}_K(n,r), \quad K \otimes_{\mathbb{Z}} A^{\mathrm{s}}_{\mathbb{Z}}(n) \cong A^{\mathrm{s}}_K(n).$$

Since $A^{\mathrm{s}}_K(n)$ has been recognized to be the coordinate ring of $\mathrm{SpM}_K(n) = \overline{\mathrm{GSp}_K(n)}$ we may interpret the spectrum of the ring $A^{\mathrm{s}}_{\mathbb{Z}}(n)$ as an integral monoid scheme $\mathrm{SpM}_{\mathbb{Z}}(n)$. Accordingly, an integral form for the symplectic Schur algebra can be obtained as the dual algebra

$$S^{\mathrm{s}}_{\mathbb{Z}}(n,r) := \mathrm{Hom}_{\mathbb{Z}}(A^{\mathrm{s}}_{\mathbb{Z}}(n,r), \mathbb{Z})$$

of its homogenous summands. By theorem 4.3 and 6.1 this is stable under base canges, that is, tensoring by a field K gives the symplectic Schur algebra $S^{\mathrm{s}}_K(n,r) = S_0(n,r) = \mathrm{Hom}_K(A^{\mathrm{s}}_K(n,r), K)$ defined over that field. An integral form for symplectic Schur algebras exists, as well, by S. Donkin's work on *generalized Schur algebras* (see [Do2]). But his approach is quite different using the theory of Lie algebras in particular the *Kostant $\mathbb{Z}$-form.* In both cases the notion of symplectic Schur algebras can be extended to more general integral domains R instead of $\mathbb{Z}$ leading to identical concepts. In our case this is $S^{\mathrm{s}}_R(n,r) := \mathrm{Hom}_R(A^{\mathrm{s}}_R(n,r), R)$. By theorem 3.3 we conclude

Corollary 6.3 *Over any noetherian integral domain R the symplectic Schur algebra is isomorphic to the centralizer algebra of the Brauer algebra $D_r(-n)$.*

For a field of characteristic zero this has been proved by S. Doty, too ([Dt] corollary 9.3. (c)). It should be remarked, that the basis dual to $\mathbf{B}_r$ together with the anti-involution defined by matrix transposition give a *cell datum* for the symplectic Schur algebra in the sense of J. Graham and G. Lehrer (cf. [Oe], 4.2.5). Thus, its representation theory can be developed easily to the extent of the treatment of *cellular algebras* in [GL].

7 Proof of theorem 6.1

Let us first reduce to showing that $\mathbf{B}_r$ is a set of generators for the R-module $A^{\mathrm{s}}_R(n,r)$. This can be done by the following proposition where notations from the proof of corollary 6.2 are used.

Proposition 7.1 $|\mathbf{B}_r| = \dim_{\mathbb{C}}(A_0(n,r))$.

Proof: We use [Do2] p. 74 ff. First the reader may check that our definition of $A_0(n)$ and $A_0(n,r)$ is identical to the one given there. According to [Do2] and 2.2c in [Do1] we have

$$\dim_{\mathbb{C}}(A_0(n,r)) = \sum_{\lambda \in \pi_0(n,r)} \dim_{\mathbb{C}}(Y_0(\lambda))^2 \tag{16}$$

where $Y_0(\lambda) := \mathrm{Ind}_{B_0}^{\mathrm{GSp}_{\mathbb{C}}(n)}(\mathbb{C}_\lambda)$ is the irreducible $\mathrm{GSp}_{\mathbb{C}}(n)$ module induced from the linear character $\mathbb{C}_\lambda$ of the Borel subgroup B_0 (notations taken from [Do2]). Here λ runs through the set $\pi_0(n,r)$ of dominant weights corresponding to the irreducibles occuring in $V^{\otimes r}$. If T_0 denotes the maximal torus of $\mathrm{GSp}_{\mathbb{C}}(n)$ we may consider the weights λ as the grouplike elements in its coordinate ring. More precisely $\lambda \in \pi_0(n,r)$ is of the form

$$\lambda = x_{11}^{\mu_1} x_{22}^{\mu_2} \ldots x_{mm}^{\mu_m} d^l$$

as can be seen from the argumentation in [Do2]. Here, $\mu := (\mu_1, \ldots, \mu_m) \in \Lambda^+(m, r-2l)$ is a partition of $r-2l$ in not more than m parts and $0 \le l \le \frac{r}{2}$ an integer. Restricting to the symplectic group $\mathrm{Sp}_{\mathbb{C}}(n)$ we have to set the coefficient of dilation d equal to 1. Thus the restriction of λ to the maximal torus of $\mathrm{Sp}_{\mathbb{C}}(n)$ is just the dominant weight

$$\bar{\lambda} = x_{11}^{\mu_1} x_{22}^{\mu_2} \ldots x_{mm}^{\mu_m}$$

for the symplectic group itself. Furthermore, it is easy to show that restricting the $\mathrm{GSp}_{\mathbb{C}}(n)$-module structure of $Y_0(\lambda)$ to the symplectic group gives the module $\bar{Y}(\bar{\lambda}) := \mathrm{Ind}_{\bar{B}}^{\mathrm{Sp}_{\mathbb{C}}(n)}(\mathbb{C}_{\bar{\lambda}})$ induced from the linear character $\mathbb{C}_{\bar{\lambda}}$ of the Borel subgroup $\bar{B} = B_0 \cap \mathrm{Sp}_{\mathbb{C}}(n)$ of $\mathrm{Sp}_{\mathbb{C}}(n)$ (for details see [Oe], 3.3.3). But the dimension of the latter one is known to be the cardinality of I_μ^{sym} (see [Do3] theorem 2.3 b for instance). Thus, we obtain

$$\dim_{\mathbb{C}}(A_0(n,r)) = \sum_{0 \le l \le \frac{r}{2}} \sum_{\mu \in \Lambda^+(m, r-2l)} |I_\mu^{\mathrm{sym}}|^2 = |\mathbf{B}_r|.$$

□

Observe that by theorem 4.1 the proof of 6.1 can be reduced to the case $R = \mathbb{Z}$, since the definition of bideterminants over R and $\mathbb{Z}$ respectively commutes with the isomorphism μ_R when R is considered as a $\mathbb{Z}$-algebra. Now, suppose we have shown that $\mathbf{B}_r$ generates $A_{\mathbb{Z}}^{\mathrm{s}}(n,r) \subseteq A_{\mathbb{C}}^{\mathrm{s}}(n,r)$ as a $\mathbb{Z}$-module. Then the image of $\mathbf{B}_r$ in $A_0(n,r)$ under the epimorphism considered in the proof of 6.2 is a set of generators, too. By the above proposition it must be a basis of $A_0(n,r)$. Consequentely, there can't be any relations ammong the elements of $\mathbf{B}_r$, especially none with integer coefficients, giving the desired result.

The proof that $\mathbf{B}_r$ is indeed a set of generators will follow from a symplectic version of the famous *straightening formula*. For convenience of the reader we will first state the algorithm leading to the classical straightening formula. To do so, we put an order on the set $\Lambda^+(r)$ of partitions of r writing $\lambda < \mu$ if the dual λ' occurs before the dual μ' in the lexicographical order. In this order the fundamental weight $\omega_r := (1, 1, \ldots, 1) \in \Lambda^+(r,r)$ is the largest element, whereas $\alpha_r := (r) \in \Lambda^+(1,r)$ is the smallest one. We abbreviate $A := A_R(n,r)$ and define $A(> \lambda)$ resp. $A(\ge \lambda)$ to be the R-linear span in A of all bideterminants $T^\mu(\mathbf{i} : \mathbf{j})$ such that $\mu > \lambda$ resp. $\mu \ge \lambda$. For $\lambda = \omega_r$ we set $A(> \omega_r) := 0$. Clearly $A = A(\ge \alpha_r)$.

Proposition 7.2 (Classical Straightening Algorithm) *Let $\lambda \in \Lambda^+(r)$ be a partition of r and $\mathbf{j} \in I(n,r)\backslash I_\lambda$. Then to each $\mathbf{k} \in I(n,r)$ satisfying $\mathbf{k} = \mathbf{j}w$* [2] *for some $w \in \mathcal{S}_r$ and $\mathbf{k} \ll \mathbf{j}$ there is an element $a_{\mathbf{jk}} \in R$ such that in A the following congruence relation holds for all $\mathbf{i} \in I(n,r)$:*

$$T^\lambda(\mathbf{i}:\mathbf{j}) \equiv \sum_{\mathbf{k} \ll \mathbf{j}} a_{\mathbf{jk}} T^\lambda(\mathbf{i}:\mathbf{k}) \quad mod \quad A(> \lambda).$$

Here, the order on $I(n,r)$ is the lexicographical one according to the given order on $\underline{n}$, in our case $\ll$. A proof of the proposition can be found for example in [Mr], 2.5.7.

Let us now state the symplectic analogon. First consider the algebra

$$A_R^{\mathrm{sh}}(n) := A_R^{\mathrm{s}}(n) / \langle d \rangle$$

where $\langle d \rangle$ is the ideal in $A_R^{\mathrm{s}}(n)$ generated by the coefficient of dilation. It is graded since d is a homogenous element but not a bialgebra because an augmentation map is missing. In fact, it turns out that (in the case where $R = K$ is an algebraically closed field) it is the coordinate ring of the semigroup $\mathrm{SpH}_K(n) := \mathrm{SpM}_K(n) \backslash \mathrm{GSp}_K(n)$ of noninvertible elements in the symplectic monoid (see remark 7.5). Let us abbreviate its submodule of homogenous elements of degree r by $A' := A_R^{\mathrm{sh}}(n,r)$ and define $A'(> \lambda)$ and $A'(\geq \lambda)$ in the same manner as above. Further, define a map $f : I(n,r) \to \mathbb{N}_0^m$ by $f(\mathbf{i}) = (a_1, \ldots, a_m)$, where

$$a_l := |\{j \in \underline{r} | \ i_j = l \ \text{ or } \ i_j = l'\}|,$$

and order $\mathbb{N}_0^m$ writing $(a_1, \ldots, a_m) < (b_1, \ldots, b_m)$ if and only if $(b_m, b_{m-1}, \ldots, b_1)$ appears before $(a_m, a_{m-1}, \ldots, a_1)$ in the lexicographical order. Next, we obtain an order $\lhd$ on $\mathbb{N}_0^m \times I(n,r)$ in a lexicographical way, as well:

$$(a, \mathbf{i}) \lhd (b, \mathbf{j}) :\Longleftrightarrow a < b \text{ or } (a = b \text{ and } \mathbf{i} \ll \mathbf{j}).$$

Finally, this gives a new order $\lhd$ on $I(n,r)$ via the embedding $I(n,r) \hookrightarrow \mathbb{N}_0^m \times I(n,r)$ given by $\mathbf{i} \mapsto (f(\mathbf{i}), \mathbf{i})$. Now we are able to state the symplectic straightening algorithm:

Proposition 7.3 (Symplectic Straightening Algorithm) *Let $\lambda \in \Lambda^+(r)$ be a partition of r and $\mathbf{j} \in I(n,r)\backslash I_\lambda^{\mathrm{sym}}$. Then to each $\mathbf{k} \in I(n,r)$ satisfying $\mathbf{k} \lhd \mathbf{j}$ there is an element $a_{\mathbf{jk}} \in R$ such that in A' the following congruence relation holds for all $\mathbf{i} \in I(n,r)$:*

$$T^\lambda(\mathbf{i}:\mathbf{j}) \equiv \sum_{\mathbf{k} \lhd \mathbf{j}} a_{\mathbf{jk}} T^\lambda(\mathbf{i}:\mathbf{k}) \quad mod \quad A'(> \lambda).$$

Before proving this, let us deduces that $\mathbf{B}_r$ is a set of generators for $A_R^{\mathrm{s}}(n,r)$. First note that multiplication by the coefficient of dilation d leads to an exact sequence for $r > 1$

$$A_R^{\mathrm{s}}(n, r-2) \xrightarrow{\cdot d} A_R^{\mathrm{s}}(n,r) \to A_R^{\mathrm{sh}}(n,r) \to 0.$$

Therefore, by induction on r we can reduce to showing that

[2] this is missing in the statement of 2.5.7 in [Mr] but can be seen directly from the proof given there

$$\{T^\lambda(\mathbf{i}:\mathbf{j})|\ \lambda \in \Lambda^+(m,r),\ \mathbf{i},\mathbf{j} \in I_\lambda^{\mathrm{sym}}\}$$

is a set of generators for $A' = A_R^{\mathrm{sh}}(n,r)$. For this claim it is enought to show that

$$\{T^\lambda(\mathbf{i}:\mathbf{j})|\ \ \mathbf{i},\mathbf{j} \in I_\lambda^{\mathrm{sym}}\}$$

is a set of generators of $A'(\geq \lambda)/A'(> \lambda)$ for each partition λ. This can be deduced from the straightening algorithm 7.3: Since $I(n,r)$ is a finite set, the elemination of multi-indices $\mathbf{j}$ not being λ-symplectic standard in an expression $T^\lambda(\mathbf{i}:\mathbf{j})$ must terminate. This gives the straightening formula concerning the righthandside argument of $T^\lambda(\mathbf{i}:\mathbf{j})$:

Corollary 7.4 (Symplectic Straightening Formula) *Let $\lambda \in \Lambda^+(r)$ be a partition of r and $\mathbf{j} \in I(n,r)$. Then, to each $\mathbf{k} \in I_\lambda^{\mathrm{sym}}$ there is an element $a_{\mathbf{jk}} \in R$, such that in A' we have for all $\mathbf{i} \in I(n,r)$:*

$$T^\lambda(\mathbf{i}:\mathbf{j}) \equiv \sum_{\mathbf{k}\in I_\lambda^{\mathrm{sym}}} a_{\mathbf{jk}} T^\lambda(\mathbf{i}:\mathbf{k}) \quad \mathit{mod}\ A'(> \lambda).$$

Now, there is an algebra automorphism θ on $A_R(n)$ induced by matrix transposition and given by $\theta(x_{ij}) = x_{ji}$ on generators. Of course it is an anti-automorphism of coalgebras. It can be readily seen that $\theta(\mathcal{F}) = \mathcal{F}$ and $\theta(d) = d$. Therefore, it factors to an automorphism of $A_R^{\mathrm{sh}}(n)$ which will be denoted by the same symbol. From the definition of bideterminants we see $\theta(T^\lambda(\mathbf{i}:\mathbf{j})) = T^\lambda(\mathbf{j}:\mathbf{i})$.

Applying θ to the congruence relation in 7.4 we see that a non λ-symplectic standard entry $\mathbf{i}$ on the lefthandside entry of a bideterminant can be eleminated, too, not effecting the righthandside entry. Thus, it must be possible to write $T^\lambda(\mathbf{i}:\mathbf{j})$ as a sum of bideterminants $T^\lambda(\mathbf{k}:\mathbf{l})$ modulo $A'(> \lambda)$ where $\mathbf{k},\mathbf{l} \in I_\lambda^{\mathrm{sym}}$. Therefore, the proof of theorem 6.1 is finished as soon as proposition 7.3 is established.

Remark 7.5 *A symplectic straightening formula similar to 7.4 can be obtained as a special case of a theorem by de Concini ([Co], theorem 2.4) where $m = 2r$ (in the notation taken from there). The algebra denoted A in that paper becomes the coordinate ring of the semigroup denoted $\mathrm{SpH}_K(n)$ above. But, note that this result is not strong enough for our purpose since we don't know wether $A = A_K^{\mathrm{sh}}(n)$ or not. On the other hand, the latter identity follows from theorem 6.1 in a similar way to corollary 6.2 using theorem 3.6 of [Co].*

Also, the symplectic straightening formula is related to the treatment of symplectic Schur-modules in [Do3] and [Ia], section 6, as can be see from the proof of lemma 8.1 below. Concerning the latter paper it should be noted that the algebra $A_K^{sp}(\overline{n})$ defined there as the coordinate ring of the symplectic group itself is only filtered by $\sum_{t=0}^r A_K^{sp}(\overline{n},t)$ but not graded by $A_K^{sp}(\overline{n},r)$. In fact, it can be deduced from the above remark that $A_K^{\mathrm{sh}}(n)$ is the corresponding graded algebra, that is $A_K^{sp}(\overline{n},r) = A_K^{\mathrm{sh}}(n,r)$ as K-vector spaces.

8 Proof of proposition 7.3

First, some considerations about the exterior algebra

$$\bigwedge_R(n) := \mathcal{T}(V)/\left\langle v_i \otimes v_j + v_j \otimes v_i, v_k \otimes v_k;\ i,j,k \in \underline{n}\right\rangle$$

are needed. It is easy to see and well known that this graded R-algebra is a *comodule algebra* for the bialgebra $A_R(n)$, i.e. it is an $A_R(n)$-comodule such that multiplication is a morphism of comodules. Infact, the homogenous summands $\bigwedge_R(n,r)$ of $\bigwedge_R(n)$ are comodules for the coalgebras $A_R(n,r)$. Since $A_R^{\mathrm{s}}(n)$ is an epimorphic image of $A_R(n)$ the exterior algebra is a comodule algebra for the latter one, as well.

As usual we write $v_i \wedge v_j$ for the residue class of $v_i \otimes v_j$ in $\bigwedge_R(n)$ and denote arbitrary multiplications by $\wedge$, too. For a subset $I := \{i_1, \ldots, i_r\} \subseteq \underline{n}$ ordered $i_1 \ll i_2 \ll \ldots \ll i_r$ by the given order $\ll$ on $\underline{n}$ (called an ordered subset in the sequel) we use the abbriviation $v_I := v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_r}$. The elements v_I give a basis of $\bigwedge_R(n)$ if I ranges over all subsets of $\underline{n}$. A basis for $\bigwedge_R(n,r)$ is obtained when I ranges over all subsets of cardinality r, the collection of which will be denoted by $P(n,r)$. The comodule structure of $\bigwedge_R(n,r)$ can be described in a simple way using bideterminants for the partition $\omega_r := (1^r) = (1,1,\ldots,1)$. These are just the usual $r \times r$-minor determinants. Denoting the structure map by $\tau_\wedge : \bigwedge_R(n) \to \bigwedge_R(n) \otimes A_R(n)$ (we will use the same symbol in the case $A_R^{\mathrm{s}}(n)$ later on) we explicitely have

$$\tau_\wedge(v_J) = \sum_{I \in P(n,r)} v_I \otimes T^{\omega_r}(\mathbf{i} : \mathbf{j}) \tag{17}$$

where $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{j} = (j_1, \ldots, j_r)$ are the multi-indices corresponding to the ordered subsets $I := \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots, j_r\}$, respectively. Define $d_k := v_{k'} \wedge v_k$ and $d_K := d_{k_1} \wedge d_{k_2} \wedge \ldots \wedge d_{k_a}$ for a subset $K := \{k_1, \ldots, k_a\} \subseteq \underline{m}$ of cardinality a. Again, we write $P(m,a)$ for the collection of all such subsets K. Note that the d_k are in the center of the exterior algebra and in particular commute with each other. Thus d_K is defined independent on the order of the elements of K. Set

$$D_a := \sum_{K \in P(m,a)} d_K$$

and let N be the ideal in $\bigwedge_R(n)$ generated by the elements $D_1, D_2, \ldots, D_m$. One crucial point in the proof of proposition 7.3 is to show that the elements D_a are invariant under the bialgebra $A_R^{\mathrm{s}}(n)$. But first, let us establish a prototype straightening algorithm inside the graded algebra $\bigwedge_R^{\mathrm{s}}(n)$ with respect to the set $I_{\omega_r}^{\mathrm{sym}}$. We call an ordered subset $I \in P(n,r)$ *symplectic* if the corresponding multi-index $\mathbf{i}$ is ω_r-symplectic standard.

Lemma 8.1 *Let $I \in P(n,r)$ be non symplectic. Then, to each $J \in P(n,r)$ such that the inequallity $f(\mathbf{j}) < f(\mathbf{i})$ holds for corresponding multi-indices $\mathbf{i}$ and $\mathbf{j}$ there is $a_{IJ} \in R$, such that in $\bigwedge_R(n)$ the following congruence relation holds:*

$$v_I \equiv \sum_{J \in P(n,r),\ f(\mathbf{j}) < f(\mathbf{i})} a_{IJ} v_J \quad mod \quad N$$

PROOF: We follow [Do3]. Clearly, we may reduce to the case $R = \mathbb{Z}$ since $\bigwedge_R(n) \cong R \otimes_{\mathbb{Z}} \bigwedge_{\mathbb{Z}}(n)$ and the canonical isomorphism therein respects the basis elements v_I and the ideal N. If $K \subseteq \underline{m}$ then in $\bigwedge_R(n)$ we calculate as in [Do3], 2.2.

$$\Big(\sum_{k \in K} d_k\Big)^a = a! \sum_{L \in P(m,a),\ L \subseteq K} d_L \tag{18}$$

In the case $K = \underline{m}$ this implies $D_1^a = a! D_a$. Set $x := \sum_{k \in K} d_k$ and $y := D_1 - x$. It follows from (18) that the equation

$$x^a = (-1)^a y^a + \sum_{b=1}^{a} \binom{a}{b} (-y)^{a-b} b! D_b$$

is divisible by $a!$. Setting $X_a^K := \sum_{L \in P(m,a),\ L \subseteq K} d_L$ and $M := \underline{m} \backslash K$ this yields

$$X_a^K = (-1)^a X_a^M + \sum_{b=1}^{a} (-1)^{a-b} X_{a-b}^M D_b$$

since $\bigwedge_{\mathbb{Z}}(n)$ is a free $\mathbb{Z}$-module. We obtain $X_a^K \equiv (-1)^a X_a^M$ modulo the ideal N and in the special case $K \in P(m,a)$

$$d_K \equiv (-1)^a \sum_{L \in P(m,a),\ L \cap K = \emptyset} d_L \mod N. \tag{19}$$

The proof can be finished now in a similar way to the proof of the *Symplectic Carter-Lusztig Lemma* in [Do3]: Since I is not symplectic there is a number $s \in \underline{r}$ such that $I = \{i_1, \ldots, i_r\}$ contains an element i_s satisfying $i_s < s$ if $i_s \leq m$ or $i_s' < s$ if $i_s > m$. We assume s to be as small as possible with this property. Let $k, l \in \underline{m}$ be the unique numbers with $\{k, k'\} = \{i_s, i_s'\}$ and $\{l, l'\} = \{i_{s-1}, i_{s-1}'\}$. By minimality of s we have $l \geq s-1$. On the other hand, $i_{s-1} \ll i_s$ implies $l \leq k$. This gives $s - 1 \leq l \leq k < s$, that is $k = l$ and $s = k+1$.

Now let $I_s := \{i_1, \ldots, i_s\}$ be the ordered subset of the first s entries of I and K be the set of all $p \in \underline{m}$ such that both p and p' are contained in I_s. Setting $K' := \{p' |\ p \in K\}$ and $H := I \backslash (K \cup K')$ we obtain $v_I = v_H d_K$ since the elements d_p are in the center of $\bigwedge_{\mathbb{Z}}(n)$. Therefore, by equation (19) it remains to show that for all subsets $L \in P(m,a)$ which do not intersect K we have $f(\mathbf{j}) < f(\mathbf{i})$ where $\mathbf{j}$ is the multi-index corresponding to the ordered set $J := H \cup L \cup L'$. Since $v_J = v_H d_L = 0$ if L or $L' := \{p' |\ p \in L\}$ contains an element of H we further may assume $H \cap L = H \cap L' = \emptyset$. Now, suppose $L \subseteq \underline{k}$. Let G be the set of all $g \in \underline{k} \backslash K$ such that g or g' lies in I_s. Since $I_s \backslash (K \cup K') \subseteq H$ the intersection of L and G is empty, as well. This gives a contradiction

$$k + 1 = s = |I_s| = 2|K| + |G| = |K| + |L| + |G| = |K \cup L \cup G| \leq k$$

for K, L and G are disjoint subsets of $\underline{k}$ by assumption on L. Thus, the largest element t of L must be greater than k, whereas all elements of K are smaller or equal to k. If $f(\mathbf{i}) = (b_1, \ldots, b_m)$ and $f(\mathbf{j}) = (a_1, \ldots, a_m)$ it follows that $a_t = b_t + 2 > b_t$ and $a_l = b_l$ for $t < l \leq m$. By definition of our order on $\mathbb{N}_0^m$ this means $f(\mathbf{j}) < f(\mathbf{i})$ completing the proof. □

As mentioned before, the crucial point in the proof of proposition 7.3 is contained in the following

Proposition 8.2 *The elements D_a are invariant under the bialgebra $A^{\mathrm{s}}_R(n)$. The corresponding grouplike elements in $A^{\mathrm{s}}_R(n)$ are the a-th powers of the coefficient of dilation, more precisely:*

$$\tau_\wedge(D_a) = D_a \otimes d^a \quad \in \quad \bigwedge\nolimits_R(n, 2a) \otimes A^{\mathrm{s}}_R(n, 2a)$$

Proof: In the case $a = 1$ this easily follows from (15) using the relations given by the set F defined in (14). If we could divide by $a!$, we would be able to finish the proof right now using $D_1^a = a!D_a$ and the fact that multiplication is a morphism of $A^{\mathrm{s}}_R(n)$-comodules. But, as this is not possible in general (note, that we don't know if $A^{\mathrm{s}}_{\mathbb{Z}}(n)$ is a free $\mathbb{Z}$-module, yet) we have to proceed in another way. We set $r = 2a$ and

$$G_{\mathbf{i}} := \sum_{L \in P(m,a)} T^{\omega_r}(\mathbf{i} : \mathbf{j}(L))$$

where $\mathbf{j}(L)$ is the multi-index corresponding to the ordered subset $L \cup L' = \{l, l' |\ l \in L\}$ of $\underline{n}$. If $\mathbf{i}$ is the multi-index corresponding to the ordered subset $I \in P(n, r)$ we also write $G_I = G_{\mathbf{i}}$. By (17) we have

$$\tau_\wedge(D_a) = \sum_{I \in P(n,r)} v_I \otimes G_I.$$

Therefore, it remains to show that $G_I = d^a$ if there is a $K \in P(m, a)$ such that $I = K \cup K'$ and $G_I = 0$ otherwise.

If $\mathbf{i} = (i_1, \ldots, i_r)$ is a multi-index and $l \leq a$ we set $\mathbf{i}^l := (i_{2l-1}, i_{2l}) \in I(n, 2)$. Since the result is clear in the case $a = 1$ we know

$$G_{\mathbf{i}^l} = \begin{cases} 0 & i'_{2l-1} \neq i_{2l} \\ d & i'_{2l-1} = i_{2l} \leq m \\ -d & i'_{2l-1} = i_{2l} > m \end{cases} .$$

Therefore, the element $G'_{\mathbf{i}} := \prod_{l=1}^a G_{\mathbf{i}^l}$ is zero or $\pm d^a$ depending on $\mathbf{i}$. Let us investigate this in more detail. If $\sigma \in \mathcal{S}_r$ denotes the involution defined by $\sigma(2l-1) = 2l, \sigma(2l) = 2l-1$ for $l \in \underline{a}$ and W the centralizer of σ in $\mathcal{S}_r$, then clearly $G'_{\mathbf{i}} = 0$ if and only if $G'_{\mathbf{i}w} = 0$ for all $w \in W$. The group W is isomorphic to the *Weyl group* of type C_a. It is a semi direct product of the normal subgroup W^1 consisting of all $\pi \in W$ which permute neighboured pairs together with W^0, the subgroup generated by the transpositions $(2l-1, 2l)$ for $l \in \underline{a}$. W^1 is isomorphic to $\mathcal{S}_a$, whereas the group W^0 can be identified with $(\mathbb{Z}/2\mathbb{Z})^a$. Choose a set H of left coset representatives for W in $\mathcal{S}_r$ the element representing W itsself being $\mathrm{id}_{\underline{r}}$.

Now, if $\mathbf{i}$ corresponds to an ordered set $I = K \cup K'$ for some $K \in P(a, n)$ the inequality $G'_{\mathbf{i}\pi} \neq 0$ holds for a permutation $\pi \in \mathcal{S}_r$ if and only if $\pi \in W$. Thus, for $h \in H$ we have $G'_{\mathbf{i}h} = 0$ if $h \neq \mathrm{id}$ and $G'_{\mathbf{i}} = d^a$. If there is no K such that $I = K \cup K'$ one clearly has $G'_{\mathbf{i}h} = 0$ for al $h \in H$. Therefore, the proof is finished as soon as we have shown

$$G_{\mathbf{i}} = \sum_{h \in H} \mathrm{sign}(h) G'_{\mathbf{i}h}. \tag{20}$$

To this claim let $\mu := (a, a) \in \Lambda^+(2, r)$ be the partition of r whose diagram consists of two rows of lenght a. To a multi-index $\mathbf{l} = (l_1, l_2, \ldots, l_a) \in I(m, a)$ another multi-index

$\mathbf{j}(\mathbf{l}) := (l'_1, l_1, l'_2, l_2, \ldots, l'_a, l_a) \in I(n,r)$ can be associated. Using this notation and reading $T^\mu(\mathbf{i} : \mathbf{j}(\mathbf{l}))$ as a product of a 2×2-determinants we obtain the formula

$$G'_{\mathbf{i}} = \sum_{\mathbf{l} \in I(m,a)} T^\mu(\mathbf{i} : \mathbf{j}(\mathbf{l})) \quad \in A^{\mathrm{s}}_R(n,r).$$

On the other hand, using *Laplace Duality* (see for example [Mr] 2.5.1) we calculate

$$T^{\omega_r}(\mathbf{i} : \mathbf{j}) = \sum_{h \in H,\, \pi \in W^1} \operatorname{sign}(h) T^\mu(\mathbf{i}h\pi : \mathbf{j}) = \sum_{h \in H} \operatorname{sign}(h) \sum_{\pi \in W^1} T^\mu(\mathbf{i}h : \mathbf{j}\pi). \tag{21}$$

Therein, note that W^0 is precisely the column stabilizer of the basic tableaux $T^\mu_{\mathbf{b}}$, whereas the column stabilizer of $T^{\omega_r}_{\mathbf{b}}$ is all of $\mathcal{S}_r$ (here $\mathbf{b} := (1', 1, 2', 2, \ldots, a', a)$). Also, note that HW^1 is a set of left coset representatives of W^0 and that all permutations of W^1 are even. Furthermore, in the righthand side equation we have used the commutativity between the 2×2-determinant factors of $T^\mu(\mathbf{i}h\pi : \mathbf{j}) = T^\mu(\mathbf{i}h : \mathbf{j}\pi^{-1})$. Now, the proof of (20) can be reduced to the verification of

$$\sum_{L \in P(m,r)} \sum_{\pi \in W^1} \sum_{h \in H} \operatorname{sign}(h) T^\mu(\mathbf{i}h : \mathbf{j}(L)\pi) = \sum_{\mathbf{l} \in I(m,a)} \sum_{h \in H} \operatorname{sign}(h) T^\mu(\mathbf{i}h : \mathbf{j}(\mathbf{l})) \tag{22}$$

To this claim we associate to a multi-index $\mathbf{l} \in I(m,a)$ its contents $|\mathbf{l}| = \lambda = (\lambda_1, \ldots, \lambda_m)$ which is defined by $\lambda_i := |\{1 \leq t \leq a |\ l_t = i\}|$. It is a composition of a into m parts, that is an m-tuple of non negative integers λ_i summing up to a. These compositions count the set of $\mathcal{S}_a$-orbits in $I(m,a)$. Denoting the set of all such compositions by $\Lambda(m,a)$ we can write down the righthand term in (22) as a sum of subsums $\sum_{\lambda \in \Lambda(m,r)} \Sigma_\lambda$ each of which is given by

$$\Sigma_\lambda := \sum_{|\mathbf{l}| = \lambda} \sum_{h \in H} \operatorname{sign}(h) T^\mu(\mathbf{i}h : \mathbf{j}(\mathbf{l}))$$

Now, the subsum Σ_{ω_a} (for $\omega_a = (1^a)$) is just the lefthand side in (22). Therefore, it remains to show that all other subsums are zero. To this claim we denote the cardinality of the standard Young subgroup $\mathcal{S}_\lambda$ of $\mathcal{S}_a$ corresponding to the composition $\lambda \in \Lambda(m,a)$ by $k_\lambda := |\mathcal{S}_\lambda|$. If $\mathbf{k} = (k_1, \ldots, k_a) \in I(m,a)$ is the unique multi-index with contents λ and $k_1 \leq k_2 \leq \ldots \leq k_r$ (the *initial index* corresponding to λ) then $\mathcal{S}_\lambda$ is just the stabilizer of $\mathbf{k}$ in $\mathcal{S}_a$. Identifying $\mathcal{S}_a$ with W^1 it is the stabilizer of $\mathbf{j}(\mathbf{k}) \in I(n,r)$ in W^1. Applying (21) again we obtain

$$T^{\omega_r}(\mathbf{i} : \mathbf{j}(\mathbf{k})) = \sum_{h \in H} \operatorname{sign}(h) \sum_{\pi \in W^1} T^\mu(\mathbf{i}h : \mathbf{j}(\mathbf{k})\pi) =$$

$$k_\lambda \sum_{h \in H} \operatorname{sign}(h) \sum_{|\mathbf{l}| = \lambda} T^\mu(\mathbf{i}h : \mathbf{j}(\mathbf{l})) = k_\lambda \Sigma_\lambda.$$

Now, $T^{\omega_r}(\mathbf{i} : \mathbf{j}(\mathbf{k}))$ must be zero since $|\mathbf{k}| \neq \omega_a$ implies that $\mathbf{k}$ contains at least one number twice. We obtain $k_\lambda \Sigma_\lambda = 0$ and because this equation is allready valid in the free $\mathbb{Z}$-module $A_{\mathbb{Z}}(n,r)$ we conclude $\Sigma_\lambda = 0$ for all $\lambda \neq \omega_a$ completing the proof. □

Let us prove proposition 7.3 in the case $\lambda = \omega_r$ first. Take $\mathbf{j} \in I(n,r) \backslash I^{\mathrm{sym}}_{\omega_r}$. Using the classical straighening algorithm 7.2 we may assume $\mathbf{j} \in I_{\omega_r} \backslash I^{\mathrm{sym}}_{\omega_r}$ (observe that $\mathbf{k} =$

$\mathbf{j}w$ implies $f(\mathbf{k}) = f(\mathbf{j})$). This means, that $\mathbf{j}$ is a multi-index corresponding to a non symplectic ordered set $J \in P(n,r)$ in the sense of lemma 8.1. Application of the latter one yields

$$X := v_J - \sum_{K \in P(n,r),\ f(\mathbf{k}) < f(\mathbf{j})} a_{\mathbf{jk}} v_K \quad \in N$$

According to proposition 8.2 $\tau_\wedge(X)$ must be contained in $\bigwedge_R(n) \otimes < d >$ where $< d >$ denotes the ideal in $A^{\mathrm{s}}_R(n)$ generated by d. Applying (17) we obtain the following equation in $\bigwedge_R(n,r) \otimes A^{\mathrm{sh}}_R(n,r)$:

$$\sum_{I \in P(n,r)} v_I \otimes \left(T^{\omega_r}(\mathbf{i}:\mathbf{j}) - \sum_{K \in P(n,r),\ \mathbf{k} \lhd \mathbf{j}} a_{\mathbf{jk}} T^{\omega_r}(\mathbf{i}:\mathbf{k}) \right) = 0.$$

Since $\{v_I |\ I \in P(n,r)\}$ is a basis of $\bigwedge_R(n,r)$ each individual summand in the summation over $P(n,r)$ must be zero. This gives the desired result in the case of multi-indices $\mathbf{i}$ corresponding to ordered subsets $I \in P(n,r)$, that is $\mathbf{i} \in I_{\omega_r}$. The general case for $\mathbf{i}$ can be deduced from this, easily (see [Oe], 3.11.4).

Now, lets turn to the general case of λ. Again, we may assume $\mathbf{j} \in I_\lambda \backslash I^{\mathrm{sym}}_\lambda$ by the classical straightening algorithm. Let $\lambda' = (\mu_1, \dots, \mu_p)$ be the dual partition ($p = \lambda_1$). We spilt $\mathbf{j}$ into p multi-indices $\mathbf{j}^l \in I(n, \mu_l)$ where for each $l \in \underline{p}$ the entries of $\mathbf{j}^l$ are taken from the l-th column of $T^\lambda_{\mathbf{j}}$. The same thing can be done with $\mathbf{i}$. Since $\mathbf{j}$ is not λ-symplectic standard but standard there must be a colunm s such that $\mathbf{j}^s$ is not ω_{μ_s}-symplectic standard. Applying the result to the known case of $T^{\omega_{\mu_s}}(\mathbf{i}^s : \mathbf{j}^s)$ we obtain

$$T^\lambda(\mathbf{i}:\mathbf{j}) = T^{\omega_{\mu_1}}(\mathbf{i}^1 : \mathbf{j}^1) T^{\omega_{\mu_2}}(\mathbf{i}^2 : \mathbf{j}^2) \dots T^{\omega_{\mu_s}}(\mathbf{i}^s : \mathbf{j}^s) \dots T^{\omega_{\mu_p}}(\mathbf{i}^p : \mathbf{j}^p)$$

$$\equiv \sum a_{\mathbf{j}^s\mathbf{k}^s} T^{\omega_{\mu_1}}(\mathbf{i}^1 : \mathbf{j}^1) \dots T^{\omega_{\mu_s}}(\mathbf{i}^s : \mathbf{k}^s) \dots T^{\omega_{\mu_p}}(\mathbf{i}^p : \mathbf{j}^p) \quad = \quad \sum a_{\mathbf{jk}} T^\lambda(\mathbf{i}:\mathbf{k}).$$

Therein, $\mathbf{k}^s \in I(n, \mu_s)$ satisfies $\mathbf{k}^s \lhd \mathbf{j}^s$, $\mathbf{k} \in I(n,r)$ is constructed from $\mathbf{j}$ replacing the entries of $\mathbf{j}^s$ by that of $\mathbf{k}^s$ and $a_{\mathbf{jk}}$ is the same as $a_{\mathbf{j}^s\mathbf{k}^s}$ for the corresponding $\mathbf{k}^s$. One easily checks $\mathbf{k} \lhd \mathbf{j}$ and the proof of 7.3 is completed.

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