Global Decomposition Numbers for Cellular Algebras

preliminary version, february 1997

Sebastian Oehms, Universität Stuttgart [1]

1 Introduction

We introduce a concept of global decomposition numbers for cellular algebras A over a principal ideal domain R. Such algebras have been defined by Graham and Lehrer in [GL]. In the category of R-finitely generated A-modules, you always have a set of *standard* resp. *costandard* modules, the former of which Graham and Lehrer called *cell representations* and which we will denote by $\triangle(\lambda)$ resp. $\bigtriangledown(\lambda)$ (for the indicated connections with *quasi-hereditary* algebras we refer to [KX]). Here λ runs through a poset Λ which comes along with the definition of A.

Now, if there is an R-algebra structure on a field K, i.e. a ring homomorphism from R to K, it is shown in [GL] that there is a unique maximal $A_K := (K \otimes_R A)$-submodule in $\triangle_K(\lambda) := K \otimes_R \triangle(\lambda)$ (not necessarily proper) and that the corresponding nonzero simple quotients $L_K(\lambda)$ are absolutely irreducible and give all the irreducibles A_K-modules. In this situation you can define *decomposition numbers* $d_K^{\lambda\mu}$ as the integer coefficients of $\triangle_K(\lambda)$ in the *Grothendieck group* of A_K with respect to the basis element $L_K(\mu)$.

In this paper we are going to define analogues of $L_K(\mu)$ and $d_K^{\lambda\mu}$ with respect to the algebra A over the original ground ring R, such that you can get the above described simple modules and decomposition numbers by specialising. As examples for A we will treat the group algebras of the symmetric groups $\mathcal{S}_3$ and $\mathcal{S}_4$ over the ring $\mathbb{Z}$. In this case, the standard modules $\triangle(\lambda)$ are known as *Specht modules* and the index set Λ consists of partitions of n. It turns out that the global decomposition numbers contain informations on the module category of $\mathbb{Z}\mathcal{S}_4$, which vanishes in all specialisations, i.e. which can not be reconstructed from the knowledge of ordinary decomposition numbers for all characteristics.

This work is purely conceptual, the proofs being almost trivial. But anyway, it provides a new view and calculus for decomposition numbers. Since more information survives in this calculus, there might be a chance to understand things better.

2 General Concepts

Let R be an arbitrary unital commutative ring and A a unital R-algebra, i.e. a unital ring A together with a ring homomorphism from R into the center of A sending 1 to 1. We denote by mod_R the category of finitely generated R-modules and by mod_A the category of right A-modules which are finitely generated as R-modules. Assume that the isomorphism classes of mod_R and mod_A form sets $\mathbb{S}_R$ and $\mathbb{S}_A$. The monoidal structure of the category mod_R – coming along with the tensorproduct over R – induces the structure of a commutative monoid on $\mathbb{S}_R$. Thus the multiplication of two elements $u, v \in \mathbb{S}_R$ given

[1] E-Mail: seba@mathematik.uni-stuttgart.de

by representatives $U, V \in \mathrm{mod}_R$ is defined as the isomorphism class of $U \otimes_R V$. The unit element 1 is the isomorphism class of R. and in addition you have an element 0 corresponding to the zero module. In a similar manner there is an action of the monoid $\mathbb{S}_R$ from the left on the set $\mathbb{S}_A$, since the tensor product $U \otimes_R M$ of an R-module U with a right A-module gives again a right A-module.

Now let $\underset{\sim}{\widetilde{R}} := \mathbb{Z}\mathbb{S}_R$ be the monoid algebra on $\mathbb{S}_R$ over the integers and $\widetilde{\underline{A}}$ the free abelian group on $\mathbb{S}_A$. The above action of $\mathbb{S}_R$ on $\mathbb{S}_A$ induces a $\widetilde{\underline{R}}$-module structure on $\widetilde{\underline{A}}$. Let I_R be the abelian subgroup of $\widetilde{\underline{R}}$ generated be the expressions $U' - U + U''$ for all $U, U', U'' \in \mathbb{S}_R$, such that there is a short split exact sequence (from now on we will not deitinguish between elements in $\mathbb{S}_R$ resp. $\mathbb{S}_A$ and representatives for these elements in mod_R resp. mod_A).

$$0 \to U' \to U \to U'' \to 0.$$

Since tensoring split exact sequences yields again split exact sequences, I_R is obviously an ideal in $\widetilde{\underline{R}}$. We denote the residue class ring by

$$\underline{R} := \widetilde{\underline{R}}/I_R.$$

Similarly, in $\widetilde{\underline{A}}$ we consider the abelian subgroup I_A generated by all expressions $M' - M + M''$, such that there is a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of right A-modules which is split as a sequence of R-modules. It is obviously an $\widetilde{\underline{R}}$-submodule of $\widetilde{\underline{A}}$ and clearly one has

$$I_R I_A \subseteq I_A.$$

Therefore the quotient

$$\underline{A} := \widetilde{\underline{A}}/I_A$$

can be considered as an $\underline{R}$-module. Before proceeding let us give some examples. For this purpose denote by $\mathbb{Y}_R \subseteq \mathbb{S}_R$ the set of isomorphism classes corresponding to the indecomposable R-modules and by $\mathbb{X}_R \subseteq \mathbb{S}_R$ the set corresponding to cyclic R-modules, i.e. modules isomorphic to R/I for some ideal I in R. Note that $\mathbb{X}_R$ is a submonoid of $\mathbb{S}_R$ and both contain 1 and 0. Therefore, the $\mathbb{Z}$-linear span of $\mathbb{X}_R$ in $\underline{R}$ is a subring.

Now if R is noetherian, then the ring $\underline{R}$ is generated by the residue classes of $\mathbb{Y}_R$. If in addition the Krull-Schmidt theorem holds in mod_R, that is, if decomposition into indecomposables is unique, then $\underline{R}$ is free abelian on $\mathbb{Y}_R \backslash \{0\}$.

Let us look in more detail to the case where R is a principal ideal domain. Here you have $\mathbb{Y}_R \subseteq \mathbb{X}_R$ and the former is a submonoid as well. More precisely, for each pair (p, l) of a prime $p \in R$ and a positive integer $l \in \mathbb{N}$ you have an element in $\mathbb{Y}_R$ whose representative is $R/(p^l)$ and which we denote by $[p^l]$. From the main theorem on modules over principal ideal domains it follows that for different pairs $(p, l) \neq (q, k)$ you get different elements $[p^l] \neq [q^k]$ and that all elements of $\mathbb{Y}_R$ – except 1 and 0 – are of this form. The multiplication in $\mathbb{Y}_R$ for elements other than 1 and 0 is given by the rule

$$[p^l][q^k] = \begin{cases} 0 & \text{if } p \neq q \\ [p^{\min(l,k)}] & \text{if } p = q \end{cases} . \tag{1}$$

Now $\underline{R}$ is canonically isomorphic to the integral monoid algebra of $\mathbb{Y}_R$ factored by the one dimensional span of the element $0 \in \mathbb{Y}_R$. In the case R is a principal ideal domain we will always identify $\underline{R}$ in this way and use the notation $[p^l]$ for basis elements as above. In the same manner we write $[n]$ for the residue class of the isomorphism type of $R/(n)$ if $n \in R$ is arbitrary. Thus $[n] = [p_1^{l_1}] + [p_2^{l_2}] + \ldots + [p_k^{l_k}]$ if $n = p_1^{l_1} p_2^{l_2} \ldots p_k^{l_k}$ is the primedecomposition of n in R. In the case where R is a field, $\underline{R}$ is canonically isomorphic to $\mathbb{Z}$ and $\underline{A}$ is the *Grothendieck group* $\mathrm{G}_0(A)$ of A considered as a $\mathbb{Z}$-module.

We call a right A-module $M \in \mathrm{mod}_A$ *reducible by R-sums* iff M contains an A-submodule N which is a direct summand of R-modules in M. If M is not reducible by R sums we call it *irreducible by R sums*. The concept is clearly invariant under isomorphisms of A-modules. Let $\mathbb{I}_A$ be the subset of $\mathbb{S}_A$ consisting of isomorphism classes of by R sums irreducible A-modules. If R is noetherian it is easily seen that $\underline{A}$ is generated by $\mathbb{I}_A$ as a $\mathbb{Z}$-module.

3 Base Changes

Let S be another commutative ring which might be considered as an R-algebra via some ring homomorphism $\varphi : R \to S$. In this situation there is a functor

$$\varphi := S \otimes_R - : \mathrm{mod}_R \to \mathrm{mod}_S$$

leading to a map $\mathbb{S}_\varphi : \mathbb{S}_R \to \mathbb{S}_S$ of monoids, given by $\mathbb{S}_\varphi(U) = V$ iff $V \cong S \otimes_R U$. In the second step this induces a ring homomorphism $\underline{\widetilde{\varphi}} : \underline{\widetilde{R}} \to \underline{\widetilde{S}}$ carrying the ideal I_R into the ideal I_S, since tensoring short split exact sequences yields again short split exact sequences. Thus we end up with a ring homomorphism

$$\underline{\varphi} : \underline{R} \to \underline{S}.$$

In the case where S is a field, this is a ring homomorphism to the integers given on a residue class $[M]$ in $\underline{R}$ for some $M \in \mathbb{S}_R$ by

$$\underline{\varphi}([M]) = \dim_S(S \otimes_R M).$$

For a principal ideal domain you therefore have $\underline{\varphi}(1) = 1$ and

$$\underline{\varphi}([p^l]) = \begin{cases} 0 & \text{if } (p) \neq \ker(R \to S) \\ 1 & \text{if } (p) = \ker(R \to S) \end{cases} . \tag{2}$$

Turning to the algebra A we write $A_S := S \otimes_R A$ for the base extended algebra. As above, there is a functor

$$\psi := S \otimes_R - : \mathrm{mod}_A \to \mathrm{mod}_{A_S}$$

leading to a map $\mathbb{S}_\psi : \mathbb{S}_A \to \mathbb{S}_{A_S}$. This induces a $\mathbb{Z}$-module homomorphism $\underline{\widetilde{\psi}} : \underline{\widetilde{A}} \to \underline{\widetilde{A_S}}$ and since $\underline{\widetilde{\psi}}(I_A) \subseteq I_{A_S}$ finally a $\mathbb{Z}$-module homomorphism

$$\underline{\psi} : \underline{A} \to \underline{A_S}.$$

Concerning the action of $\mathbb{S}_R$ on $\mathbb{S}_A$ one obviously has $\mathbb{S}_\psi(UM) = \mathbb{S}_\varphi(U)\mathbb{S}_\psi(M)$ for all $U \in \mathbb{S}_R$ and $M \in \mathbb{S}_A$ which leads to

$$\underline{\psi}(ra) = \underline{\varphi}(r)\underline{\psi}(a) \quad \text{for all } r \in \underline{R},\ a \in \underline{A}$$

If we interpret $\underline{S}$ as an $\underline{R}$-algebra via $\underline{\varphi}$, we can restate this as

Proposition 3.1 *There is an $\underline{S}$-module homomorphism*

$$\underline{\psi}_S : \underline{S} \otimes_{\underline{R}} \underline{A} \to \underline{A_S}$$

given by $\underline{\psi}_S(s \otimes a) := s\underline{\psi}(a)$ for all $s \in \underline{S}$ and $a \in \underline{A}$.

Now, on the other hand there are functors

$$\varphi' : \mathrm{mod}_S \to \mathrm{mod}_R, \quad \psi' : \mathrm{mod}_{A_S} \to \mathrm{mod}_A$$

carrying an S-module resp. A_S-module M to the same abelian group, but inflating the action of S resp. A_S to an action of R resp. A on M. Going again through the above procedure, one obtains maps of abelian groups

$$\underline{\varphi'} : \underline{S} \to \underline{R}, \quad \underline{\psi'} : \underline{A_S} \to \underline{A}.$$

If the map $R \to S$ is surjective, then you have $\varphi \circ \varphi'(U) \cong U$ and $\psi \circ \psi'(M) \cong M$ for all S-modules U and A_S-modules M. Therefore in this case, one obtains

$$\underline{\varphi} \circ \underline{\varphi'} = \mathrm{id}_{\underline{S}} \quad \text{and} \quad \underline{\psi} \circ \underline{\psi'} = \mathrm{id}_{\underline{A_S}}.$$

In the case where $R \to S$ is surjektiv, there corresponds an element in $\mathbb{X}_R$ to S, i.e. the isomorphism class of the R-module S. This leads to a corresponding element $[S]$ in $\underline{R}$. Denote the left multiplication by $[S]$ in $\underline{A}$ by l_S.

Proposition 3.2 *Suppose that the R-algebra structure map $R \to S$ on S is surjective. Then $\underline{\psi'}$ gives an embedding of $\underline{A_S}$ into $\underline{A}$ as a direct summand, and you have*

$$\underline{\psi'} \circ \underline{\psi} = l_S.$$

Furthermore, the $\underline{S}$-module homomorphism $\underline{\psi}_S$ of Proposition 3.1 is an isomorphism with inverse given by $\underline{\psi}_S^{-1}(a) := 1_{\underline{S}} \otimes \underline{\psi'}(a)$.

Proof: The first two statements are clear from definitions and what has been said before (note that $[S]$ is an idempotent and therefore l_S a projection map of $\underline{R}$-modules). For the third one, it is enough to show that $\kappa(a) := 1_{\underline{S}} \otimes \underline{\psi'}(a)$ defines an inverse as a map of $\mathbb{Z}$-modules. Now $\underline{\psi}_S \circ \kappa = \mathrm{id}$ is directly clear from the above righthand formula, whereas $\kappa \circ \underline{\psi}_S = \mathrm{id}$ holds by surjectivity of $\underline{\varphi}$ following from the lefthand formula. □

The set $\mathbb{Y}_R \cap \mathbb{X}_R$ contains the set of fields F which are epimorphic images of R and which we denote by $\mathbb{F}_R$. These are precisely the elements of $\mathbb{X}_R$ corresponding to simple R-modules $F = R/I$, one for each maximal ideal I of R. For any $F \in \mathbb{F}_R$ the $\mathbb{Z}$-module $\underline{A_F}$ is the Grothendieck group of A_F as mentioned before. Therefore the proposition shows, that all such Grothendieck groups are embedded in $\underline{A}$ as direct summands and that a projection on them is given by multiplication with the idempotent $[F] \in \underline{R}$. We set

$$I^0 := \{a \in \underline{A} |\ [F]a = 0\ \forall F \in \mathbb{F}_R\}.$$

As an intersection of the kernels of the $\underline{R}$-module homomorphisms l_F this is obviously a submodule. To see that for an R-algebra S the map $\underline{\psi}$ takes I^0 to the corresponding submodule I^0_S of $\underline{A_S}$, one uses the fact that the R-algebra structure map $\rho : R \to S$ induces an injective map $\tau : \mathbb{F}_S \to \mathbb{F}_R$ on a representative $F = S/I$ given by $\tau(F) := [R/\rho^{-1}(I)]$. It is easily seen that the equation $[\tau(F)]a = [F]\underline{\psi}(a)$ holds for all $F \in \mathbb{F}_S$ and $a \in \underline{A}$. Thus, setting

$$\underline{A}^0 := \underline{A}/I^0$$

it follows that $\underline{\psi}$ factors to a map $\underline{\psi}^0 : \underline{A}^0 \to \underline{A_S}^0$. In the case where S is a field you have $\mathbb{F}_S = \{1\}$ and therefore $I^0_S = \{0\}$. This means that $\underline{A}^0$ still maps to the Grothendieck group $\mathrm{G}_0(A_S)$ for any field S that is an R-algebra.

4 Residual Series

As we have seen, factoring out the submodule I^0 we don't cut off more infomation as is contained in the collection of all Grothendieck groups for fields. However, we can save additional informations, that is, factoring out some proper submodule of I^0 instead of I^0 if it remains possible to define global decomposition numbers in the corresponding quotient of $\underline{A}$. We now give the definition of such a one.

For each isomorphism class $C \in \mathbb{X}_R$ of cyclic R-modules there is a well defined map $\mathrm{res}_C : \mathbb{S}_R \to \mathbb{S}_R$ to each $U \in \mathbb{S}_R$ given by the isomorphism class of the kernel of the map

$$U \to (R/I) \otimes_R U, \quad u \mapsto 1 \otimes u,$$

where I is the ideal in R such that R/I is of isomorphism type C. Thus $\mathrm{res}_C(U)$ is just the isomorphism class of the R-modul IU. It is easily seen that res_C maps short split exact sequences to short split exact sequences, such that it induces an additive map on $\underline{R}$ which we denote by the same symbol. Now, if we take for U a right A-module M, then of course $\mathrm{res}_C(M)$ is again a right A-module. Therefore one obtains a similar additive map $\mathrm{res}_C : \underline{A} \to \underline{A}$ for which we will use the same symbol as well and call it the *C-residuum map*. We write res^i_C for iterating res_C i times and define

$$N^i := \{a \in \underline{A} |\ \mathrm{res}^i_F(a) \in I^0\ \ \forall F \in \mathbb{F}_R\}.$$

These are of course abelian subgroups of $\underline{A}$ as the intersection of preimages of I^0 under the additive maps res^i_F. Later on, we will see that they in fact are $\underline{R}$-submodules in the case where R is a principle ideal domain. But in general this is not true.

We now define inductively a descending chain $I^0 \supseteq I^1 \supseteq \ldots$ of $\underline{R}$-submodules. The definition of I^0 has been given. For $i > 1$ let I^i be the submodule of I^{i-1} generated by $I^{i-1} \cap N^i$. Set

$$I^\infty := \bigcap_{i=0}^{\infty} I^i \quad \text{and} \quad \underline{A}^i := \underline{A}/I^i, \quad \underline{A}^\infty := \underline{A}/I^\infty.$$

Let us illustrate this in the case of a principle ideal domain. Then $F \in \mathbb{F}_R$ is of the form $F = R/(p)$ for a prime p, that is, $[F] = [p]$ in the above introduced notation. We write $\mathrm{res}_p := \mathrm{res}_F$ for short. Since $\underline{R}$ is generated by $\mathbb{Y}_R$ as a $\mathbb{Z}$-module we only need to calculate for prime powers q^l

$$\mathrm{res}_p([q^l]) = \begin{cases} [q^l] & \text{if } p \neq q \\ [q^{l-1}] & \text{if } p = q \quad \text{and} \quad l > 1 \\ 0 & \text{if } p = q \quad \text{and} \quad l = 1 \end{cases} . \tag{3}$$

The p-residuum of 1 and 0 is obviously 1 resp. 0. This, together with (1) shows that res_p is multiplicative, i.e. $\mathrm{res}_p(ra) = \mathrm{res}_p(r)\mathrm{res}_p(a)$ for $r, a \in \underline{R}$ resp. $a \in \underline{A}$. Therefore, the N^i are submodules in $\underline{A}$ as mentioned above, and it follows $I^i = I^{i-1} \cap N^i$. This means that an element $a \in \underline{A}$ is zero in $\underline{A}^\infty$, iff for all $i \in \mathbb{N}_0$ and all primes $p, q \in R$ you have $[q]\mathrm{res}_p^i(a) = 0$. Here res_p^0 has to be defined as the identity map.

From (1) and (3) it is easy to see that $[q]\mathrm{res}_p(a) = [q]a$ if $q \neq p$. Therefore, $[p]\mathrm{res}_p^i(a) = 0$ implies $[q]\mathrm{res}_p^i(a) = 0$ for all primes q if a is assumed to lie in I^0. This shows

Lemma 4.1 *Let R be a principal ideal domain. Two elements a, b of $\underline{A}$ coincide in $\underline{A}^\infty$ iff they have identical* residual series *for all primes p. The latter one is for $x \in \underline{A}$ defined by*

$$\mathrm{Res}_p(x) := ([p]\mathrm{res}_p^0(x), [p]\mathrm{res}_p^1(x), [p]\mathrm{res}_p^2(x), \ldots)$$

A similar statement holds with respect to $\underline{A}^i$ and the first i entries of both series.

It should be remarked that res_F is not multiplicative in general. As an example, take the ring $R := \mathbb{Z}[x, x^{-1}]$ of Laurent polynomials in the indeterminant x which is important in the context of Hecke algebras, q-Schur algebras, Birman-Murakami-Wenzl algebras, etc. Let $F := R/(p, x-1) \cong GF(p)$ be the Galois field to the prime $p \in \mathbb{Z}$ with R-module structure given by the natural projection. In the same manner, $U := R/(x-1) \cong \mathbb{Z}$ and $V := R/(p) \cong GF(p)[x, x^{-1}]$ are considered as R-modules. Then you have $\mathrm{res}_F(U) \cong U$, $\mathrm{res}_F(V) \cong V$, and since $U \otimes_R V \cong F$ it follows

$$\mathrm{res}_F([U])\mathrm{res}_F([V]) = [F] \neq 0 = \mathrm{res}_F([U][V]).$$

5 Application to Cellular Algebras

We now restrict A to a special type of R-algebra which have been described using a couple of axioms by Graham and Lehrer ([GL]) and which are called cellular algebras. Instead of listing these axioms we rather explain the properties of A and mod_A which will be needed in the sequel. For more details we refer to [GL].

1. There is a finite poset Λ and for each $\lambda \in \Lambda$ a pair $\triangle(\lambda), \bigtriangledown(\lambda) \in \mathrm{mod}_A$ of A-modules called the *standard* and the *costandard* module of type λ. Both are free and isomorphic to each other as R-modules and there is an A-module homomorphism
$$f_\lambda : \triangle(\lambda) \to \bigtriangledown(\lambda).$$
2. For each commutative ring S being an R-algebra, $A_S := S \otimes_R A$ is again a cellular algebra and the functor $\psi := S \otimes_R -$ carries $\triangle(\lambda), \bigtriangledown(\lambda), f_\lambda$ to the corresponding objects and morphisms $\triangle_S(\lambda), \bigtriangledown_S(\lambda), f_{S,\lambda}$ with respect to A_S.
3. For each field K being an R-algebra, the image $L_K(\lambda)$ of $f_{K,\lambda}$ is an irreducible A_K-module as long as $f_{K,\lambda}$ is not the zero map. All irreducible A_K-modules occur in this way. Furthermore, if $\lambda \neq \mu$ and $f_{K,\lambda} \neq 0 \neq f_{K,\mu}$, then the (nesessarily absolutely) irreducible modules $L_K(\lambda)$ and $L_K(\mu)$ are non isomorphic.
4. For a field K and $\mu \in \Lambda$ such that $f_{K,\mu} \neq 0$, let $d_K^{\lambda\mu}$ be the multiplictiy of $L_K(\mu)$ as a composition factor of $\triangle_K(\lambda)$. Then $d_K^{\mu\mu} = 1$ and $d_K^{\lambda\mu} \neq 0$ implies $\lambda \leq \mu$.

Let us compare these statements with corresponding definitions and results in [GL]. We assume that A is finitely generated as an R-module. This implies the finiteness of Λ. Now, as we consider right modules, our $\triangle(\lambda)$ corresponds to $W(\lambda)^*$ in [GL]. The costandard modules are defined as their duals
$$\bigtriangledown(\lambda) := \mathrm{Hom}_R(\triangle(\lambda), R)$$
where the action of A is given by $(la)(d) := l(da^*)$ for all $l \in \mathrm{Hom}_R(\triangle(\lambda), R)$, $a \in A$ and $d \in \triangle(\lambda)$. Here $*$ denotes the anti involution on A which exists by one of the axioms. Now, on $W(\lambda)$ there is a symmetric bilinear form ϕ_λ with $\phi_\lambda(a^*x, y) = \phi_\lambda(x, ay)$. Since $W(\lambda)$ coincides with $W(\lambda)^*$ as an R-module we have a bilinear form on $\triangle(\lambda)$ with $\phi_\lambda(xa, y) = \phi_\lambda(x, ya^*)$ (since the operation on $W(\lambda)^*$ is just the left action on $W(\lambda)$ pulled to the right via $*$). This in turn leads to the A-module homomorphism f_λ being defined by $f_\lambda(x)(y) := \phi_\lambda(x, y)$.

The compatibility with the functor ψ is obvious (compare (1.8) in [GL]). The statements concerning the $L_K(\lambda)$ follow from (3.2) and (3.4) of [GL]. For this purpose, note that the radical of ϕ_λ is nothing but the kernel of f_λ where again the action of A is pulled from left to right via $*$. Finally, the statement on the decomposition numbers $d_K^{\lambda\mu}$ follows from (3.6) of [GL].

If K is a field and all $f_{K,\lambda}$ are isomorphisms, then according to Theorem (3.8) of [GL] A_K is semisimple. For the field Q of fractions on an integral domain R this is by flatness the case iff f_λ is injective for all $\lambda \in \Lambda$. If this is the case call A *generic semisimple*. Even for such an A it is possible that $f_{K,\lambda} = 0$ for some field K. For a prime p we will throughout denote by $F := [p] = R/(p)$ the corresponding residue class field for short. We set
$$\Lambda_p := \{\lambda \in \Lambda |\ f_{F,\lambda} \neq 0\}, \quad \Lambda_p^- := \Lambda \backslash \Lambda_p, \quad P_\lambda := \{p \in R |\ p \text{ prim}, \lambda \in \Lambda_p^-\}.$$
According to (3.10) in [GL] for a field K, the algebra A_K is *quasi-hereditary* if $f_{K,\lambda} \neq 0$ for all $\lambda \in \Lambda$. Let us call A *integrally quasi-hereditary* if $\Lambda_p^- = \emptyset$ for all primes $p \in R$.

We now define a global version of $L_K(\lambda)$ in $\underline{A}$. Throughout, we will not distinguish between right A-modules and their residue classes in $\underline{A}, \underline{A}^i$ or $\underline{A}^\infty$ any more. Let $T(\lambda)$ be the cokernel of f_λ. We set

$$L(\lambda) := \nabla(\lambda) - T(\lambda) \quad \in \underline{A}.$$

Note that $L(\lambda)$ is not the residue class of some A-module in general. But if K is a field and an R-algebra, then the map $\underline{\psi}$ of chapter 3 carries this element to the residue class of $L_K(\lambda)$ in the Grothendieck group $\mathrm{G}_0(A_K)$.

Let $\mathcal{L}$ be the $\underline{R}$-linear span of all $L(\lambda)$ in $\underline{A}^\infty$. We wish to show that for a generic semisimple cellular algebra A over a pid you always have $\mathcal{L} = \underline{A}^\infty$. From now on, let us assume that R is a pid. We call an element $r \in \underline{R}$ positive if in the unique expression with respect to the $\mathbb{Z}$-basis $\mathbb{Y}_R \backslash \{0\}$ of $\underline{R}$ all integer coefficients are nonnegative. Write $\underline{R}^+$ for the submonoid of all these elements and $\mathcal{L}^+$ for the $\underline{R}^+$ span of $\{L(\lambda) |\ \lambda \in \Lambda\}$ in $\mathcal{L}$. Note that $x \in \underline{R}^+$ is zero, iff $[p]x = 0$ for all primes p, whereas $x \in \underline{R}$ is zero, iff the residual series $\mathrm{Res}_p(x) := ([p]\mathrm{res}_p^i(x), [p]\mathrm{res}_p^1(x), \ldots)$ in $\underline{R}$ is zero for all primes p.

Lemma 5.1 *Let A be cellular and M a right A-module. If M is an R-torsion module, then you have $M \in \mathcal{L}^+$ as elements of $\underline{A}^\infty$.*

PROOF: Since the R-annihilator of $x \in M$ is contained in the R-annihilator of xa for all $a \in A$, it follows that the R-module decomposition of M into primary components is in fact a decomposition of A-modules. Thus we may assume that M is primary as an R-module. Let p be the corresponding prime. Now for all primes $q \neq p$ the residual series $\mathrm{Res}_q(M)$ is constant zero. Let us consider $\mathrm{Res}_p(M)$. Recall that $F := R/(p)$ and $\lambda \in \Lambda_p$ means $f_{F,\lambda} \neq 0$, that is $L_F(\lambda) = [p]L(\lambda) \neq 0$. Since these simple A_F-modules form a basis of the Grothendieck group of A_F, there are unique nonnegative integers k_λ^i such that

$$[p]\mathrm{res}_p^i(M) = \sum_{\lambda \in \Lambda_p} k_\lambda^i [p] L(\lambda).$$

For all $l \in \mathbb{N}$ there is an A_F-epimorphism

$$[p]\mathrm{res}_p^{l-1}(M) = p^{l-1}M/p^l M \to p^l M/p^{l+1}M = [p]\mathrm{res}_p^l(M)$$

given by multiplication by p. This shows that $k_\lambda^i - k_\lambda^{i+1}$ is nonnegative. There is a smallest number j, such that $\mathrm{res}_p^l(M) = 0$ for all $l \geq j$, since M is a torsion module. We set

$$N := \sum_{\lambda \in \Lambda_p} (\sum_{i=1}^{j} (k_\lambda^{i-1} - k_\lambda^i)[p^i]) L(\lambda).$$

This is obviously contained in $\mathcal{L}^+$. By Lemma 4.1 it remains to show that $\mathrm{Res}_q(M) = \mathrm{Res}_q(N)$ for all primes q. For $q \neq p$ both series are identically zero. For $q = p$ by multiplicativity of res_p^i, it is enough to show

$$[p]\mathrm{res}_p^i(\sum_{i=1}^{j} (k_\lambda^{i-1} - k_\lambda^i)[p^i]) = k_\lambda^i [p]$$

for all i. This can be calculated with help of (3). $\square$

Corollary 5.2 *For a generic semisimple cellular algebra A all costandard modules $\nabla(\lambda)$ are contained in $\mathcal{L}^+$ as elements of $\underline{A}^\infty$.*

PROOF: This is immediate from the definition of $L(\lambda)$ and the Lemma 5.1 since under the above assumption $T(\lambda)$ is an R-torsion module. □

Lemma 5.3 *Let Q be the field of fractions on R, M and N two full A-lattices in the same A_Q-module V. Then in $\underline{A}^\infty$ the identity $M = N$ holds.*

PROOF: By Lemma 4.1 we have to show that the residual series of M and N coincide for all primes p. The epimorphism from $[p]\mathrm{res}_p^i(M)$ to $[p]\mathrm{res}_p^{i+1}(M)$ considered in the proof of Lemma 5.1 and being induced by multiplication by p must be an isomorphism in the case of a free module. Therefore, the residual series of M and N are constant. Thus the proof is finished as soon as we have shown $[p]M = [p]N$ for all primes p. Since the sum $M + N$ in V again is a full lattice,we can reduce to the case $M \subseteq N$. Now let R_p be the localisation of R at the prime ideal (p) and $M' := R_pM$, $N' := R_pN$ the corresponding R_p lattices in V. Multiplication by $[p]$ – the residue class in $\underline{R_p}$ of the field $F = R_P/(p)$ – is defined in $\underline{A_{R_p}}$ as well and leads to the same elements $[p]M' = [p]M$ resp. $[p]N' = [p]N$ in the Grothendieck group $\mathrm{G}_0(A_F)$ (see Propositions 3.1 and 3.2). Finishing the proof now is standard (cf. [CR], 16.16). □

Applying Lemma 5.3 to the image of f_λ and $\nabla(\lambda)$ as lattices in $\nabla_Q(\lambda)$, one immediately gets

Corollary 5.4 *For a generic semisimple cellular algebra A corresponding standard and costandard modules $\triangle(\lambda)$ and $\nabla(\lambda)$ coincide in $\underline{A}^\infty$ for all $\lambda \in \Lambda$.*

Theorem 5.5 *Let A be a generic semisimple cellular algebra. Then each right A-module $M \in \mathrm{mod}_A$ is contained in $\mathcal{L}^+$ as an element of $\underline{A}^\infty$. In particular $\underline{A}^\infty = \mathcal{L}$.*

PROOF: By Lemma 5.1 we can reduce to the case where M is free as an R-module, since an arbitrary M can be written $M = M_f + M_t$ with M_f free and M_t an R-torsion module (even in $\underline{A}$). Let Q be the field of fractions on R. Since A is generic semisimple, $M_Q := Q \otimes_R M$ decomposes into a direct sum of simple modules $L_Q(\lambda) = \triangle_Q(\lambda) = \nabla_Q(\lambda)$. Let $N \in \mathrm{mod}_A$ be the direct sum of the corresponding $\nabla(\lambda)$. Both M and N are full lattices in M_Q so that it follows $M = N$ in $\underline{A}^\infty$ by Lemma 5.3. Since $N \in \mathcal{L}^+$ by Corollary 5.2 the proof is finished. □

Throughout, let us now assume that A is generic semisimple. Note that in this case the set P_λ defined above is finite for all λ. Set $z_\lambda := 1 - \sum_{p \in P_\lambda}[p^{l_p}] \in \underline{R}$ where l_p is the largest number such that $[p^{l_p}]L(\lambda) = 0$.

Proposition 5.6 *There are uniquely determined elements $d^{\lambda\mu} \in \underline{R}$ assigned to each pair $\lambda, \mu \in \Lambda$ which satisfy the following conditions:*

(a) $d^{\lambda\mu} \in \underline{R}^+$ *for all* $\lambda \neq \mu$ *and* $d^{\lambda\lambda} = z_\lambda$

(b) $[p]d^{\lambda\mu} = 0$ *for all* $\mu \in \Lambda_p^-$

(c) $\nabla(\lambda) = \sum_{\mu\in\Lambda} d^{\lambda\mu} L(\mu)$ *holds in* $\underline{A}^\infty$.

PROOF: To show existence, there are elements $g^{\lambda\mu} \in \underline{R}^+$ such that $T(\lambda) = \sum_{\mu\in\Lambda} g^{\lambda\mu} L(\mu)$ by Lemma 5.1. Furthermore, the proof of this lemma shows that $[p]g^{\lambda\mu} = 0$ for all $\mu \in \Lambda_p^-$. If $\lambda \in \Lambda_p$ we multiply the equation $\nabla(\lambda) = L(\lambda)+T(\lambda)$ by $[p]$. Because $[p]\nabla(\lambda) = \nabla_F(\lambda)$ and $[p]L(\lambda) = L_F(\lambda)$, we get $[p](g^{\lambda\lambda}+1) = [p]d_F^{\lambda\lambda} = [p]$ thus $[p]g^{\lambda\lambda} = 0$. Because $g^{\lambda\lambda} \in \underline{R}^+$, this in turn implies $g^{\lambda\lambda} = 0$ for all λ. Setting $d^{\lambda\lambda} := z_\lambda$ and $d^{\lambda\mu} := g^{\lambda\mu}$ for $\lambda \neq \mu$ we are finished since $\nabla(\lambda) = L(\lambda) + T(\lambda)$ and obviously $L(\lambda) = z_\lambda L(\lambda)$.

For uniqueness, take $(d^{\lambda\mu})'$ to be a second set of such elements. Since $d^{\lambda\lambda} = (d^{\lambda\lambda})'$ by definition and for $\lambda \neq \mu$ both elemens are in $\underline{R}^+$, it is enough to show $[p]d^{\lambda\mu} = [p](d^{\lambda\mu})'$ for all primes p. In the case $\mu \in \Lambda_p^-$ this follows from condition (b). Using (b) and multiplying (c) by $[p]$ we get from Proposition 3.2 $[p]d^{\lambda\mu} = [p]d_F^{\lambda\mu} = [p](d^{\lambda\mu})'$ for all $\mu \in \Lambda_p$ where $d_F^{\lambda\mu}$ are the usual decomposition numbers for A_F. To this claim, note that $\triangle_F(\lambda) = [p]\triangle(\lambda) = [p]\nabla(\lambda) = \nabla_F(\lambda)$ in $\mathrm{G}_0(A_F)$ by Corollary 5.4 and Proposition 3.2. This completes the proof. □

For any field K you have $\underline{\varphi}(d^{\lambda\mu}) = d_K^{\lambda\mu}$ where $\underline{\varphi} : \underline{R} \to \underline{K} = \mathbb{Z}$ is the map induced by the base change to K (see chapter 3). Therefore, we call these elements the *global decomposition numbers* of A.

Proposition 5.7 *If* $\lambda \neq \mu$ *then* $d^{\lambda\mu} \neq 0$ *implies* $\lambda < \mu$.

PROOF: Since $0 \neq d^{\lambda\mu} \in \underline{R}^+$ there must be a prime $p \in R$ such that $[p]d^{\lambda\mu} \neq 0$. This implies $d_F^{\lambda\mu} \neq 0$ for the residue class field $F = R/(p)$. The corresponding known result for fields then implies the statement of the proposition. □

For any total order on Λ refining the given partial order, the matrix $D := (d^{\lambda\mu})_{\lambda,\mu\in\Lambda}$ is therefore upper triangular with respect to this total order. Setting $e^{\lambda\lambda} := 1$ and $e^{\lambda\mu} := d^{\lambda\mu}$ for $\lambda \neq \mu$ one gets a unitriangular (and so invertible) matrix $E := (e^{\lambda\mu})_{\lambda,\mu\in\Lambda}$. Note that condition (c) of proposition 5.6 holds for the $e^{\lambda\mu}$, too. We call D the *global decomposition matrix* and E the *regular decomposition matrix*. The fact that E is invertible implies that $\underline{A}^\infty$ is generated as an $\underline{R}$ module by the costandard modules $\nabla(\lambda)$, too.

Let us determine the relations among the $L(\lambda)$ resp. the $\nabla(\lambda)$. Denote by X_λ the canonical basis elements of the free $\underline{R}$-module $\underline{R}^\Lambda$. We define two epimorphisms $\sigma, \tau : \underline{R}^\Lambda \to \underline{A}^\infty$ by $\sigma(X_\lambda) := L(\lambda)$ and $\tau(X_\lambda) := \nabla(\lambda)$. If ϵ is the automorphism of $\underline{R}^\Lambda$ given by the regular decomposition matrix E, i.e. $\epsilon(X_\lambda) := \sum_{\mu\in\Lambda} e^{\lambda\mu} X_\mu$, we clearly have $\sigma\circ\epsilon = \tau$. Let $\mathcal{K}_\tau$ and $\mathcal{K}_\sigma$ be the kernels of τ and σ respectively. We only need to determine $\mathcal{K}_\tau$ since $\mathcal{K}_\sigma = \epsilon(\mathcal{K}_\tau)$.

Let $p \in R$ be a prime such that $\Lambda_p^- \neq \emptyset$. Since for all $\mu \in \Lambda_p$ we have $d^{\mu\mu} = 1$ the costandard modules $\{\nabla_F(\mu) | \ \mu \in \Lambda_p\}$ form a basis of the Grothendieck group $\mathrm{G}_0(A_F)$. Therefore there are unique integers $a_{\lambda\mu,p}$ for all $\lambda \in \Lambda_p^-$ and $\mu \in \Lambda_p$ such that

$$[p]\nabla(\lambda) = \sum_{\mu\in\Lambda_p} a_{\lambda\mu,p}[p]\nabla(\mu).$$

For each pair (λ, p) with $\lambda \in \Lambda_p^-$ we set

$$u_{\lambda,p} := X_\lambda - \sum_{\mu\in\Lambda_p} a_{\lambda\mu,p} X_\mu,$$

and let $\mathcal{K}'_\tau$ be the $\mathbb{Z}$-linear span of all elements $[p^l]u_{\lambda,p}$ for such pairs (λ,p) and positive integers $l \in \mathbb{N}$.

Proposition 5.8 *We have $\mathcal{K}_\tau = \mathcal{K}'_\tau$.*

PROOF: Since the residual series of the costandard modules are constant we have for a prime q

$$[q]\mathrm{res}^i_q(\tau([p^l]u_{\lambda,p})) = [q]\mathrm{res}^i_q([p^l])(\nabla(\lambda) - \sum_{\mu\in\Lambda_p} a_{\lambda\mu,p} \nabla(\mu)).$$

Now, $[q]\mathrm{res}^i_q([p^l]) = 0$ whenever $q \neq p$ by the formulas (1) and (3). Multiplying the bracket term by $[p]$ gives zero by construction of the numbers $a_{\lambda\mu,p}$. Therefore $\mathrm{Res}^i_q(\tau([p^l]u_{\lambda,p})) = 0$ for all primes q, thus $\mathcal{K}'_\tau \subseteq \mathcal{K}_\tau$.

Now let $x := \sum_{\lambda\in\Lambda} b_\lambda X_\lambda$ be in $\mathcal{K}_\tau$ with $b_\lambda \in \underline{R}$. We first claim that there exists $i \in \mathbb{N}_0$ such that $[p]\mathrm{res}^j_p(b_\lambda) = 0$ for all $j \geq i$, $\lambda \in \Lambda$ and primes $p \in R$. For if there is $\lambda_0 \in \Lambda$ and a prime $p \in R$ such that for all $i \in \mathbb{N}_0$ there exists $j \geq i$ giving $[p]\mathrm{res}^j_p(b_{\lambda_0}) \neq 0$, it follows that the coefficient of 1 in the unique presentation of b_{λ_0} with respect to the $\mathbb{Z}$-basis $\mathbb{Y}_R\backslash\{0\}$ of $\underline{R}$ is nonzero. Now, if Q is the field of fractions on R and $\underline{\varphi} : \underline{R} \to \underline{Q} = \mathbb{Z}$ the corresponding map of chapter 3 we must have $\underline{\varphi}(b_{\lambda_0}) \neq 0$. But this implies

$$\underline{\psi}(\tau(x)) = \sum_{\lambda\in\Lambda} \underline{\varphi}(b_\lambda) \nabla_Q (\lambda) \neq 0$$

since all costandard modules form a basis of $\mathrm{G}_0(A_Q)$. This contradict $\tau(x) = 0$ in $\underline{A}^\infty$.

We now proceed by induction on such a number i. For $i = 0$ we must have $x = 0$ since then the residual series of all b_λ are zero for all primes. Assume $i > 0$. There are uniquely determined numbers $h_{\lambda,p} \in \mathbb{Z}$ such that $[p]h_{\lambda,p} = [p]\mathrm{res}^{i-1}_p(b_\lambda)$. Set

$$y := \sum_{\lambda\in\Lambda}\sum_{p\in P_\lambda} h_{\lambda,p}[p^i]u_{\lambda,p} \quad \in \mathcal{K}'_\tau.$$

Then $z := x - y \in \mathcal{K}_\tau$. Write $z = \sum_{\lambda\in\Lambda} c_\lambda X_\lambda$. By construction of y we have $[p]\mathrm{res}^j_p(c_\lambda) = [p]\mathrm{res}^j_p(b_\lambda) = 0$ for all $j \geq i$ and all primes p. We claim $[p]\mathrm{res}^{i-1}_p(c_\lambda) = 0$ for all primes p and $\lambda \in \Lambda$. If this is shown the induction hypothesis applies to z giving $x = y + z \in \mathcal{K}'_\tau$ as required.

Keep p fixed and let $\lambda \in \Lambda^-_p$. The coefficient of X_λ in y is of the form $h_{\lambda,p}[p^i] + \sum_{q\neq p}[q^i]v_q$ with some $v_q \in \underline{R}$ being nonzero only for a finite number of primes q. Since $[p]\mathrm{res}^{i-1}_p([q^i]v_q) = [p][q^i]\mathrm{res}^{i-1}_p(v_q) = 0$, it follows $[p]\mathrm{res}^{i-1}_p(c_\lambda) = [p]\mathrm{res}^{i-1}_p(b_\lambda) - [p]h_{\lambda,p} = 0$ by definition of $h_{\lambda,p}$. Making use of $\tau(z) = 0$ we calculate

$$[p]\mathrm{res}^{i-1}_p(\tau(z)) = \sum_{\lambda\in\lambda}[p]\mathrm{res}^{i-1}_p(c_\lambda)\mathrm{res}^{i-1}_p(\nabla(\lambda)) = \sum_{\lambda\in\Lambda_p}[p]\mathrm{res}^{i-1}_p(c_\lambda)[p] \nabla (\lambda) = 0,$$

where in addition we used the fact that the residual series of $\bigtriangledown(\lambda)$ is constant. Since $\{\bigtriangledown_F(\lambda)|\ \lambda \in \Lambda_p\}$ is a basis of $\mathrm{G}_0(A_F)$ the coefficients $[p]\mathrm{res}_p^{i-1}(c_\lambda)$ must be zero for $\lambda \in \Lambda_p$ as well. $\square$

Clearly $\mathcal{K}'_\tau = 0$ iff A is integrally quasi-hereditary. Thus we have proved

Theorem 5.9 *The $\underline{R}$-module $\underline{A}^\infty$ is free iff A is integrally quasi-hereditary. If this is the case $\{L(\lambda)|\ \lambda \in \Lambda\}$ and $\{\bigtriangledown(\lambda)|\ \lambda \in \Lambda\}$ are bases of $\underline{A}^\infty$. Furthermore, the global and regular decomposition matrices coincide and the decomposition numbers are uniquely determined by condition (3) of Proposition 5.6.*

Examples of integrally quasi-hereditary generic semisimple cellular algebras are given by Schur algebras and generalizations of them.

6 Examples

As examples we will treat the $\mathbb{Z}$-group algebras $A := \mathbb{Z}\mathcal{S}_n$ of symmetric groups. Here the standard modules $\triangle(\lambda)$ are known to be the *Specht modules* ([GL], 5.7, see [JK], chapter 7 for details on Specht-modules). The reader not familiar with cellular algebras may check imediately that the conditions stated in the begining of section 5 can be realised using this modules. They are labelled by the set $\Lambda = \Lambda(n)^+$ of partitions of n and are defined as submodules of permutation modules M^λ on the cosets of parabolic subgroups corresponding to the partition $\lambda \in \Lambda$. As first example, let us look at $\lambda := \lambda_{n-1} := (n-1, 1)$. Then M^λ is the permutation representation on the cosets of the copy of $\mathcal{S}_{n-1}$ in $\mathcal{S}_n$ fixing the last element n. As a $\mathbb{Z}$-module this is free of rank n with basis $e_1, \ldots, e_n$, say. The operation of $\mathcal{S}_n$ from the right is given by $e_i\pi := e_{\pi^{-1}(i)}$. The Specht module $\triangle(\lambda)$ is the free $\mathbb{Z}$-submodule with basis $b_i := e_{i+1} - e_i$ for $i = 1, \ldots, n-1$.

The symmetric bilinear form ϕ_λ is given by restriction of the canonical bilinear form on M^λ (given by the unit matrix) to $\triangle(\lambda)$. Thus the Gram matrix of ϕ_λ is nothing but the *Cartan* matrix corresponding to the *Dynkin* diagram of type A_{n-1}. This in turn is just the coefficient matrix of $f_\lambda : \triangle(\lambda) \to \bigtriangledown(\lambda) = \mathrm{Hom}_{\mathbb{Z}}(\triangle(\lambda), \mathbb{Z})$ with respect to the basis b_i on $\triangle(\lambda)$ and the corresponding dual basis b_i^* on $\bigtriangledown(\lambda)$. The image of f_λ is consequently the span of the columns of this matrix and is easily calculated to be the span of nb_1^* and $b_i^* - ib_1^*$ for $i = 2, \ldots, n-1$. Therefore $T(\lambda)$ – the cokernel of f_λ – is generated by b_1^* and is obviously isomorphic to $\mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}$-module.

Let us show that the operation of $\mathbb{Z}\mathcal{S}_n$ on $T(\lambda)$ is trivial. We only need to check this for the long cycle $\sigma := (12\ldots n)$ or its inverse and $\tau := (12)$. On $\triangle(\lambda)$ the operation is given by $b_1\sigma = -\sum_{i=1}^{n-1} b_i$ and $b_i\sigma = b_{i-1}$ for $i > 1$ in the case of σ whereas we have $b_1\tau = -b_1$, $b_2\tau = b_1 + b_2$ and $b_i\tau = b_i$ for all $i > 2$. Transposing the corresponding coefficient matrices we get $b_1^*\sigma^{-1} = b_2^* - b_1^* = b_1^*\tau$. Since $b_2^* - b_1^*$ is congruent to b_1^* modulo the image of f_λ we are done.

The Specht module of the partition $\lambda_n := (n)$ is just the one-dimensional trivial representation with bilinear form ϕ_{λ_n} being given by the trivial Gram matrix. Therefore $\triangle(\lambda_n) = \bigtriangledown(\lambda_n) = L(\lambda_n)$ holds in $\underline{A}$. Moreover, we have shown the following equation in $\underline{A}$:

$$\bigtriangledown(\lambda_{n-1}) = L(\lambda_{n-1}) + [n]L(\lambda_n).$$

In the case of the symmetric group $\mathcal{S}_3$ there are three partitions $\lambda_1 := (1^3) = (1,1,1)$, $\lambda_2 := (2,1)$ and $\lambda_3 := (3)$. The corresponding Specht modules are free $\mathbb{Z}$-modules of rank 1, 2 and 1, respectively. The cases λ_2 and λ_3 have been treated in general above. The Specht module $\triangle(\lambda_1)$ is the sign representation and $f_\lambda(b_1) = 3!b_1^*$. Therefore we have $T(\lambda_1) = [3!] \bigtriangledown (\lambda_1) = ([2]+[3]) \bigtriangledown (\lambda_1)$. From the ordinary decomposition matrices of $\mathcal{S}_3$ for the primes 2 and 3 we get $[2] \bigtriangledown (\lambda_1) = [2]L(\lambda_3)$ and $[3] \bigtriangledown (\lambda_1) = [3]L(\lambda_3)$ leading to the global decomposition matrix of $\mathcal{S}_3$:

Global Decomposition Matrix of $\mathcal{S}_3$				
$\bigtriangledown(\lambda)$	$L(\lambda)$	λ_1 (1^3)	λ_2 $(2,1)$	λ_3 (3)
λ_1	(1^3)	$1-[6]$	$[3]$	$[2]$
λ_2	$(2,1)$		1	$[3]$
λ_3	(3)			1

Note that the corresponding equations according to condition (c) of Proposition 5.6 even hold in $\underline{A}$. This will not be true in the next example $\mathcal{S}_4$ as well as the fact that the global decomposition matrix can entirely be constructed from the knowledge of the ordinary decomposition matrices for the primes 2 and 3. On the other hand, the ordinary decomposition numbers are detected from the above by multiplying the whole table by [2] resp. [3] and then setting $[2] = 1$ resp. $[3] = 1$.

Before proceeding to the next example let us write down all the relations between the $L(\lambda)$ with the help of Proposition 5.8. We have to consider two pairs (p, λ) such that $\lambda \in \Lambda_p^-$. These are $(2, \lambda_1)$ and $(3, \lambda_1)$. We calculate $u_{\lambda_1,2} = X_{\lambda_1} - X_{\lambda_3}$ and $u_{\lambda_1,3} = X_{\lambda_1} - X_{\lambda_2} + X_{\lambda_3}$. This gives $\epsilon(u_{\lambda_1,2}) = X_{\lambda_1} + [3]X_{\lambda_2} + ([2]-1)X_{\lambda_3}$ and $\epsilon(u_{\lambda_1,3}) = X_{\lambda_1} + ([3]-1)X_{\lambda_2} + ([2]-[3]+1)X_{\lambda_3}$ leading to the following complete set of relations for $i,j \in \mathbb{N}$ and $k,l \in \mathbb{Z}$:

$$(k[2^i] + l[3^j])L(\lambda_1) = l([3^j]-[3])L(\lambda_2) + (k([2^i]-[2]) + l([3]-[3^j]))L(\lambda_3).$$

Turning to $\mathcal{S}_4$ we have 5 partitions $\lambda_1 := (1^4)$, $\lambda_2 := (2,1^2)$, $\lambda_3 := (2,2)$, $\lambda_4 := (3,1)$ and $\lambda_5 := (4)$. Let us first picture the global decomposition matrix and then comment it:

Global Decomposition Matrix of $\mathcal{S}_4$						
$\bigtriangledown(\lambda)$	$L(\lambda)$	λ_1 (1^4)	λ_2 $(2,1^2)$	λ_3 $(2,2)$	λ_4 $(3,1)$	λ_5 (4)
λ_1	(1^4)	$1-[24]$		$[3]$		$[8]$
λ_2	$(2,1^2)$		$1-[2]$		$[8]$	$[2]$
λ_3	$(2,2)$			$1-[2]$	$[2]$	$[3]$
λ_4	$(3,1)$				1	$[4]$
λ_5	(4)					1

First note that the entries [8] and [4] cannot be reconstructed from the knowledge of the ordinary decomposition matrices of $\mathcal{S}_4$. As in the preceeding example the first row follows easily from $T(\lambda_1) = [4!] \bigtriangledown (\lambda_1)$ and the knowledge of $[2] \bigtriangledown (\lambda_1) = [2]L(\lambda_5)$ and

$[3] \bigtriangledown (\lambda_1) = [3]L(\lambda_3)$. Note that $[8] \bigtriangledown (\lambda_1) = [8]L(\lambda_5)$ does not hold in $\underline{A}$ but surely in $\underline{A}^\infty$. In contrast, the equations corresponding to the rows $3, 4, 5$ hold in $\underline{A}$ the last two of which have been treated in the beginning of this chapter. We leave the calculation of $T(\lambda_3)$ as an exercise and turn to the most interesting case $T(\lambda_2)$.

Here the Specht module is a free $\mathbb{Z}$-module of rank three with basis b_1, b_2, b_3, say. The long cycle $\sigma := (1234)$ and $\tau := (12)$ operate from right by $b_1\sigma = b_1 - b_2 + b_3,\ b_2\sigma = b_1, b_3\sigma = b_2$ and $b_1\tau = -b_1,\ b_2\tau = b_3,\ b_2\tau = b_3$. The Gram matrix of the bilinear form is given by

$$2\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

and therefore the image of f_λ is spanned by $8b_1^*, 8b_2^*$ and $2(b_1^* - b_2^* + b_3^*)$. Clearly, $T(\lambda_2)$ is isomorphic to $[8] + [8] + [2]$ as a $\mathbb{Z}$-module. But anyway, as an A-module it is irreducible by $\mathbb{Z}$-sums as defined in chapter 2. Therefore it cannot be splitted in $\underline{A}$ anymore. But we can calculate in $\underline{A}^\infty$ as in the proof of Lemma 5.1. From the ordinary decomposition matrix for the prime 2, we get $[2]T(\lambda_2) = [2](L(\lambda_4) + L(\lambda_5))$ and by dimension arguments the 2-residual series must be

$$\mathrm{Res}_2(T(\lambda_2)) = ([2](L(\lambda_4) + L(\lambda_5)), [2]L(\lambda_4), [2]L(\lambda_4), 0, \ldots)$$

which completes the verification. In order to determine the relations among the $L(\lambda)$ one calculates

$$\begin{aligned}
\epsilon(u_{\lambda_1,2}) &= X_{\lambda_1} + [3]X_{\lambda_3} + ([8]-1)X_{\lambda_5} \\
\epsilon(u_{\lambda_1,3}) &= X_{\lambda_1} + ([3]-1)X_{\lambda_3} - [2]X_{\lambda_4} + ([8]-[3]+1)X_{\lambda_5} \\
\epsilon(u_{\lambda_2,2}) &= X_{\lambda_2} + ([8]-1)X_{\lambda_4} + ([2]-[4])X_{\lambda_5} \\
\epsilon(u_{\lambda_3,2}) &= X_{\lambda_3} + ([2]-1)X_{\lambda_4} + ([3]-[4]+1)X_{\lambda_5}.
\end{aligned}$$

As in the case of $\mathcal{S}_3$ a complete set of relations can be obtained from these formulas with help of Proposition 5.8.

Problems:

- Does the global decomposition matrix depend on the *cell datum* of the algebra A, i.e. on the choice of $\triangle(\lambda)$ and f_λ etc. or is it independed up to permutations on rows and columns?
- Is there an expansion of this theory to the groundring $R := \mathbb{Z}[q, q^{-1}]$ if you take additional structure on the algebra A into acount, for example informations on the open subset in $\mathrm{Spec}(R)$ corresponding to specializations for A being semisimple.

References

[CR] Curtis, C.W., Reiner, I.: Methods of Representation Theory, Vol. I, Wiley. 1981.

[GL] Graham, J.J., Lehrer, G.I.: Cellular Algebras. Invent. Math. 123 (1996), 1-34.

[JK] James, J., Kerber, A.: The Representation Theory of the Symmetric Groups. Encycl. of Math. Appl. Vol. 16, Addison Wesley, 1981.

[KX] König, S., Xi, C.: On the structure of cellular algebras. preprint.